

## Gravitationally Bound Monopoles

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We construct monopole solutions in SU(2) Einstein-Yang-Mills-Higgs theory carrying magnetic charge  $n$ . For vanishing and small Higgs self-coupling, these multimonopole solutions are gravitationally bound. Their mass *per unit charge* is lower than the mass of the  $n = 1$  monopole. For large Higgs self-coupling only a repulsive phase exists.

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*Introduction.*—Magnetic monopoles arise as topological defects in theories which undergo spontaneous symmetry breaking. In general, magnetic monopoles exist if the mapping of the vacuum manifold, associated with the symmetry breaking, onto the two-sphere is nontrivial. The existence of magnetic monopoles is consequently a generic prediction of grand unification. If magnetic monopoles were indeed present in the Universe, they would have a host of astrophysical and cosmological consequences.

Here we consider magnetic monopoles [1] and multimonopoles [2–5] in SU(2) Yang-Mills-Higgs (YMH) theory, with the Higgs field in the triplet representation. These solutions are globally regular. Since their magnetic charge is proportional to their topological charge, the regular monopole and multimonopole solutions reside in topologically nontrivial sectors of the theory. In contrast, the monopole-antimonopole pair solutions [6,7] are topologically trivial.

In the Bogomol'nyi-Prasad-Sommerfield (BPS) limit [8,9], where the strength of the Higgs self-interaction potential vanishes, the mass of the monopole and multimonopole solutions saturates its lower bound, the Bogomol'nyi bound. In particular, the mass *per unit charge* of an  $n > 1$  monopole is precisely equal to that of the  $n = 1$  monopole. There is no interaction between the monopoles in the BPS limit, because the spatial components of the stress tensor vanish [10]. The Higgs field is massless and mediates a long range attractive force which exactly cancels the long range repulsive magnetic force of the U(1) field [11,12].

For finite Higgs self-coupling, however, the Higgs field is massive and therefore decays exponentially. Consequently the long range magnetic field dominates at large distances, leading to the repulsion of like monopoles [13]. In particular, as verified numerically for  $n = 2$  and 3

monopoles [4], the mass *per unit charge* of an  $n > 1$  monopole is higher than the mass of the  $n = 1$  monopole. Thus for finite Higgs self-coupling there is only a repulsive phase between like monopoles.

Let us now consider the effect of gravity on the monopole solutions. When gravity is coupled to YMH theory, a branch of gravitating  $n = 1$  monopole solutions emerges smoothly from the flat space monopole solution [14]. The branch extends up to some maximal value of the gravitational strength, beyond which the size of the monopole core would be smaller than the Schwarzschild radius of the solution [14]. Along the branch with increasing gravitational strength, the mass of the gravitating monopole solutions decreases monotonically. Finally a degenerate horizon forms [14–17].

Similarly, in the presence of gravity a branch of gravitating monopole-antimonopole pair (MAP) solutions emerges from the flat space MAP solution [18].

In this Letter we investigate how gravity affects the static axially symmetric multimonopole solutions of SU(2) YMH theory [2–4] and whether the inclusion of gravity allows for an attractive phase of like monopoles. For a given topological charge  $n$ , we find a branch of globally regular and asymptotically flat multimonopole solutions, emerging smoothly from the corresponding flat space solution, and extending up to some maximal value of the gravitational strength. Along the branch the mass of the solutions decreases monotonically. And, indeed, we find a region of parameter space, where the mass *per unit charge* of the gravitating multimonopole solutions is lower than the mass of the gravitating  $n = 1$  monopole. Here the multimonopole solutions are gravitationally bound. Thus gravity allows for an attractive phase of like monopoles, not present in flat space.

*Ansatz.*—We consider SU(2) Einstein-Yang-Mills-Higgs (EYM) theory with action

$$S = \int \left( \frac{R}{16\pi G} - \frac{1}{2e^2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4} \text{Tr}(D_\mu\Phi D^\mu\Phi) - \frac{1}{4} \lambda \text{Tr}(\Phi^2 - \eta^2)^2 \right) \sqrt{-g} d^4x, \quad (1)$$

with Newton's constant  $G$ , Yang-Mills coupling constant  $e$ , and Higgs self-coupling constant  $\lambda$ .

In flat space SU(2) YMH possesses multimonopole solutions with axial symmetry for any topological number [2]. For topological number  $n \geq 3$ , solutions with less symmetry exist [5]. Here we extend the axially symmetric multimonopole solutions to curved space. We choose isotropic coordinates with metric [19]

$$ds^2 = -f dt^2 + \frac{m}{f} (dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2\theta d\varphi^2, \quad (2)$$

where  $f$ ,  $m$ , and  $l$  are only functions of  $r$  and  $\theta$ . The ansatz for the purely magnetic gauge field is [3,19]

$$A_\mu dx^\mu = \frac{1}{2er} [\tau_\phi^n (H_1 dr + (1 - H_2)r d\theta) - n(\tau_r^n H_3 + \tau_\theta^n (1 - H_4))r \sin\theta d\phi], \quad (3)$$

and for the Higgs field the ansatz is [3,4]

$$\Phi = (\Phi_1 \tau_r^n + \Phi_2 \tau_\theta^n). \quad (4)$$

The symbols  $\tau_r^n$ ,  $\tau_\theta^n$ , and  $\tau_\phi^n$  denote the dot products of the Cartesian vector of Pauli matrices,  $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ , with the spatial unit vectors

$$\begin{aligned} \vec{e}_r^n &= (\sin\theta \cos n\phi, \sin\theta \sin n\phi, \cos\theta), \\ \vec{e}_\theta^n &= (\cos\theta \cos n\phi, \cos\theta \sin n\phi, -\sin\theta), \\ \vec{e}_\phi^n &= (-\sin n\phi, \cos n\phi, 0), \end{aligned} \quad (5)$$

respectively. The winding number  $n$  represents the topological charge of the solutions. The four gauge field functions  $H_i$  and the two Higgs field function  $\Phi_i$  depend only on the coordinates  $r$  and  $\theta$ . For  $n = 1$  and  $H_1 = H_3 = \Phi_2 = 0$ ,  $H_2 = H_4 = K(r)$  and  $\Phi_1 = H(r)$ , the spherically symmetric ansatz of Ref. [14] is recovered.

We fix the residual gauge degree of freedom [3,19] by choosing the gauge condition  $r\partial_r H_1 - \partial_\theta H_2 = 0$  [19].

For asymptotically flat magnetically charged solutions the boundary conditions at infinity are

$$f = m = l = 1, \quad H_1 = H_2 = H_3 = H_4 = 0, \\ \Phi_1 = \eta, \quad \Phi_2 = 0. \quad (6)$$

Requiring the solutions to be regular at the origin ( $r = 0$ ) leads to the boundary conditions

$$\partial_r f = \partial_r m = \partial_r l = 0, \quad H_1 = H_3 = 0, \\ H_2 = H_4 = 1, \quad \Phi_1 = \Phi_2 = 0. \quad (7)$$

The boundary conditions along the  $\rho$  and the  $z$  axes are determined by the symmetries of the solutions. On both

axes the functions  $H_1$ ,  $H_3$ , and  $\Phi_2$  and the derivatives  $\partial_\theta f$ ,  $\partial_\theta m$ ,  $\partial_\theta l$ ,  $\partial_\theta H_2$ ,  $\partial_\theta H_4$ , and  $\partial_\theta \Phi_1$  have to vanish.

Introducing the dimensionless coordinate  $x = r\eta e$  and the Higgs field  $\phi = \Phi/\eta$ , the equations depend only on two coupling constants,  $\alpha$  and  $\beta$ ,

$$\alpha^2 = 4\pi G\eta^2, \quad \beta^2 = \frac{\lambda}{e^2}. \quad (8)$$

The mass  $M$  of the multimonopole solutions can be obtained directly from the total energy-momentum "tensor"  $\tau^{\mu\nu}$  of matter and gravitation,  $M = \int \tau^{00} d^3r$  [20], or equivalently from  $M = - \int (2T_0^0 - T_\mu^\mu) \sqrt{-g} dr d\theta d\phi$ , yielding the dimensionless mass  $\mu = \frac{e}{4\pi\eta} M$ .

*Results.*—Subject to the above boundary conditions, we solve the equations numerically [21]. We start from the flat space (multi)monopole solutions [4] and increase  $\alpha$ . Then branches of gravitating (multi)monopole solutions extend up to maximal values of  $\alpha$ ,  $\alpha_{\max}^{(n)}(\beta)$ , which increase with increasing topological charge and decrease with increasing strength of the Higgs self-coupling.

As the globally regular (multi)monopole solutions approach the critical value of  $\alpha$ , a degenerate horizon starts to form [14–16]. Indeed, the exterior of the critical solution corresponds to the exterior of an extremal Reissner-Nordström (RN) black hole with magnetic charge  $n$  [14,15,17].

In Fig. 1 we show the metric function  $f$  of the gravitating  $n = 2$  multimonomoles in the BPS limit for several values of  $\alpha$ . With increasing  $\alpha$  the function  $f$  decreases monotonically and approaches the metric function  $f^{\text{RN}} = (\frac{x/\alpha}{n+x/\alpha})^2$  of the extremal RN solution with magnetic charge  $n = 2$ . At  $\alpha = 1.5$ , its value at the origin is already very small, close to the value  $f^{\text{RN}}(0) = 0$ , representing the degenerate horizon of the extremal RN solution. Also, the angle dependence of the metric function is seen to diminish and disappear, as the critical value of  $\alpha$  is approached.

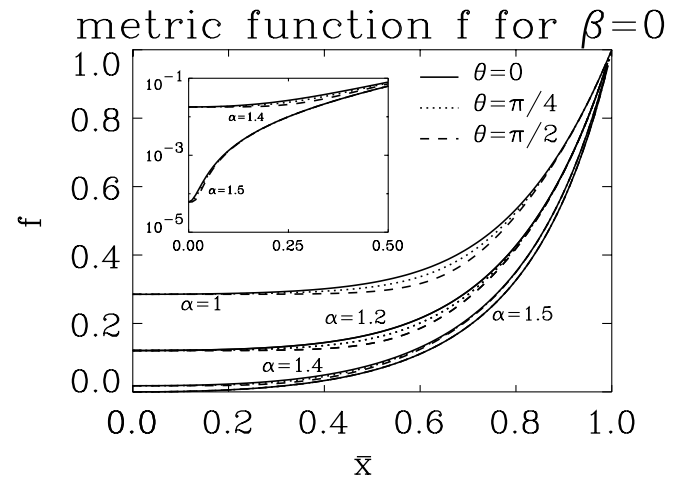


FIG. 1. The metric function  $f$  is shown as a function of the compactified dimensionless radial coordinate  $\bar{x} = x/(1+x)$  for three values of the angle  $\theta$  for the  $n = 2$  multimonomole solutions in the BPS limit, for a sequence of values of  $\alpha$ , approaching the critical value  $\alpha_{\text{cr}}$ .

Likewise, the Higgs field function  $\phi_1$ , shown in Fig. 2, approaches the value  $\phi_1^{\text{RN}} = 1$  of the RN solution [22]. The Higgs field function  $\phi_2$  and the gauge field functions approach their respective RN values as well. Satisfying Israel's theorem, the RN solution is spherically symmetric.

The mass *per unit charge* of the (multi)monopole solutions decreases with increasing  $\alpha$ . In the BPS limit, for  $\alpha = 0$  the mass *per unit charge* is precisely equal to the mass of the  $n = 1$  monopole. For  $\alpha > 0$ , however, we observe that the mass *per unit charge* of the multimono- poles is smaller than the mass of the  $n = 1$  monopole. In particular, the mass *per unit charge* decreases with increasing  $n$ . The mass *per unit charge* of  $n = 2$  and 3 multi- monopoles in the BPS limit is shown in Fig. 3. Thus in the BPS limit there is an attractive phase between like monopoles, not present in flat space. Moreover, multi- monopoles exist for gravitational coupling strength, too large for  $n = 1$  monopoles to exist.

For finite Higgs self-coupling, the flat space multi- monopoles have higher mass *per unit charge* than the  $n = 1$  monopole, allowing only for a repulsive phase between like monopoles. By continuity, this repulsive phase persists in the presence of gravity for small values of  $\alpha$ , but it can give way to an attractive phase for larger values of  $\alpha$ . Thus the repulsion between like monopoles can be overcome for small Higgs self-coupling by sufficiently strong gravitational attraction. At the equilibrium value  $\alpha_{\text{eq}}$ , multimono- pole mass *per unit charge* and monopole mass equal one another. We show the equilibrium value  $\alpha_{\text{eq}}$  in Fig. 4.  $\alpha_{\text{eq}}$  increases with increasing Higgs self-coupling, yielding a decreasing region in parameter space for the attractive phase. Finally, for large Higgs self-coupling, only a repulsive phase is left.

While singly charged monopole solutions are stable, stability of the static axially symmetric multimono- pole solutions is not obvious. We conjecture that the  $n = 2$  multimono- pole solutions are stable, as long as their mass *per unit charge* is lower than the mass of the  $n = 1$  monopole. For topological number  $n \geq 3$ , however, solutions with only discrete symmetry exist in flat space

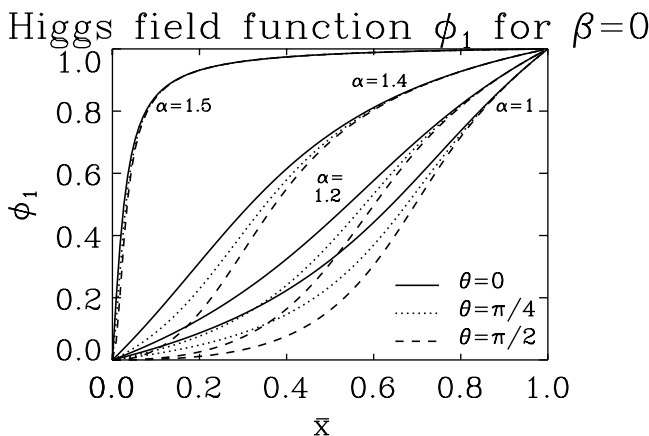


FIG. 2. Same as Fig. 1 for the Higgs field function  $\phi_1$ .

[5], which, by continuity, should also be present in curved space (at least for small gravitational strength). For a given topological number  $n > 2$ , such multimono- pole solutions without rotational symmetry may possess a lower mass than the corresponding axially symmetric solutions. The axially symmetric solutions may therefore not represent global minima in their respective topological sectors, even if their mass *per unit charge* is lower than the mass of the  $n = 1$  monopole.

**Black holes.**—Besides embedded Abelian black hole solutions, like, e.g., the magnetically charged RN solutions encountered above, SU(2) EYMH theory also possesses genuine non-Abelian black hole solutions [14]. The SU(2) EYMH black hole solutions then are no longer uniquely determined by their mass and charge (for vanishing angular momentum). The non-Abelian black hole solutions, considered as black holes within (multi)monopoles, therefore represent counterexamples to the “no-hair” theorem [14].

While static spherically symmetric ( $n = 1$ ) EYMH black holes were studied in much detail [14], non-Abelian black hole solutions with magnetic charge  $n > 1$  were previously obtained only perturbatively [23]. Replacing the boundary conditions at the origin (7) by the appropriate set of boundary conditions at some horizon radius  $x_h$  [19,21], we obtain static axially symmetric black hole solutions with magnetic charge  $n > 1$  numerically [21]. These black hole solutions are asymptotically flat, as required by the boundary conditions at infinity (6), and they possess a regular deformed horizon [19,21]. Being static and not spherically symmetric, these black hole solutions represent new counterexamples to Israel's theorem. While previous (nonperturbative) counterexamples [19,24] to Israel's theorem are not classically stable, black holes within multimono- poles should provide classically stable counterexamples [23].

Finally we note that the gravitational properties of mono- pole spacetimes near the black hole threshold lend themselves to obtain new insight into the laws of black hole

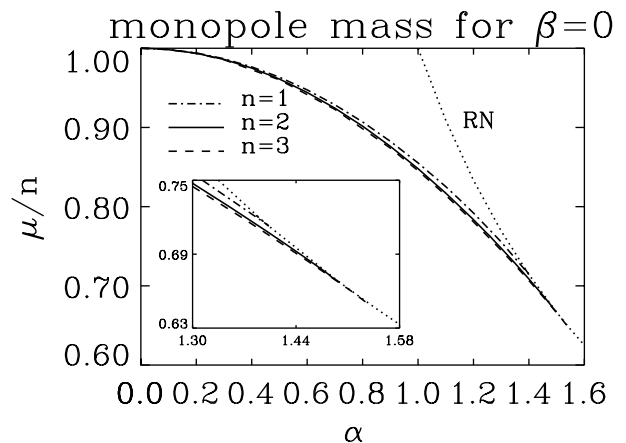


FIG. 3. The mass *per unit charge* in units of  $4\pi\eta/e$  is shown as a function of  $\alpha$  for the  $n = 2$  and 3 multimono- poles in the BPS limit. For comparison the mass of the monopole and the mass of the extremal RN solutions are also shown.

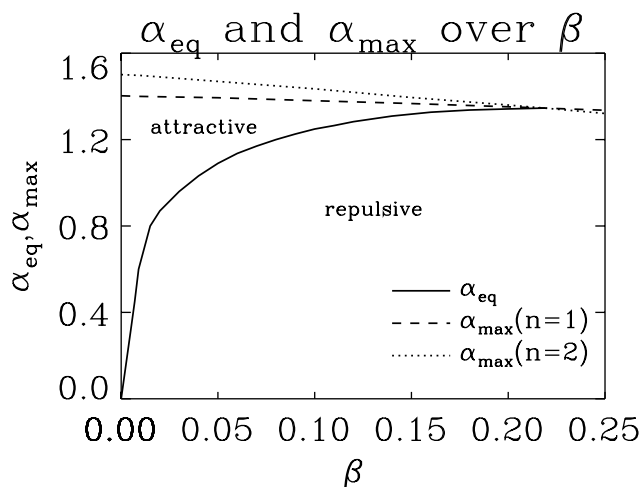


FIG. 4. The equilibrium value of  $\alpha$ ,  $\alpha_{eq}$ , where the multi-monopole mass *per unit charge* and monopole mass equal one another, is shown as a function of  $\beta$  for the  $n = 2$  multi-monopoles. Also shown are the maximal values  $\alpha_{max}$  for the  $n = 1$  and 2 (multi)monopoles.

thermodynamics [25]. Also, insight concerning the cosmic censorship hypothesis might be obtained from the static black hole solutions without spherical symmetry [26].

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