# Generalized Jordan-Wigner Transformations

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We introduce a new spin-fermion mapping, for arbitrary spin $S$ generating the SU(2) group algebra, that constitutes a natural generalization of the Jordan-Wigner transformation for $S = \frac{1}{2}$. The mapping, valid for regular lattices in any spatial dimension $d$, serves to unravel hidden symmetries. We illustrate the power of the transformation by finding exact solutions to lattice models previously unsolved by standard techniques. We also show the existence of the Haldane gap in $S = 1$ bilinear nearest-neighbor Heisenberg spin chains and discuss the relevance of the mapping to models of strongly correlated electrons. Moreover, we present a general spin-anyon mapping for the case $d \leq 2$.

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*Introduction.*—Theories of magnetism in normal matter are direct manifestations of quantum mechanics. The vast majority of phenomena occurring in magnets, such as different types of magnetically ordered states, are often simply described by interacting quantum spins [1]. Although quantum spins do not behave as either pure boson or fermion operators, different representations have invoked such particle statistics, for example, the Holstein-Primakoff, Schwinger, and Dyson-Maleev boson representations for arbitrary spin $S$, or the Jordan-Wigner (JW) and Majorana fermion representations for spin $S = \frac{1}{2}$ magnets.

It is convenient to reformulate a difficult strongly correlated problem in a way that it becomes more manageable; in some cases there is an exact dualism. This is the idea behind the bosonization techniques and the different algebra representations of a physical problem. The point is that these different representations help us understand various aspects of the same problem by transforming intricate interaction terms into simpler ones. Often, fundamental symmetries which are hidden in one representation are manifest in the other and, moreover, problems which seem untractable can even be exactly solved after the mapping. The simplest and perhaps most popular example is the equivalence between the Heisenberg-Ising $S = \frac{1}{2}$ $XXZ$ chain and a model of interacting spinless fermions through the JW transformation [2].

The JW transformation involves the $S = \frac{1}{2}$ irreducible representation of the Lie group SU(2). Here we generalize this spin-fermion mapping to any irreducible representation of dimension $2S + 1$. The three generators $S_j^\mu$ ($\mu = x, y, z$) of the Lie group for each lattice site $j$ satisfy the equal-time commutation relations [3]

$$[S_j^\mu, S_k^\nu] = i\delta_{jk}\epsilon_{\mu\nu\lambda}S_j^\lambda \,, \tag{1}$$

with $\epsilon$ the totally antisymmetric Levi-Civita symbol. The algebra generated by the (linear and Hermitian operators) $S_j^\mu$ is the enveloping algebra of the group SU(2). In terms of the ladder operators $S_j^\pm = S_j^x \pm iS_j^y$

$$\begin{aligned} &[S_j^+, S_j^-] = 2S_j^z, \qquad [S_j^z, S_j^\pm] = \pm S_j^\pm \,, \\ &\{S_j^+, S_j^-\} = 2(S(S+1) - (S_j^z)^2)\,. \end{aligned} \tag{2}$$

We start by analyzing the one-dimensional $S = 1$ case. Then, we will show a generalization to arbitrary spin and spatial dimension $d$.

$S = 1$ *mapping.*—We introduce the following composite operators $f_j^\dagger = \bar{c}_{j1}^\dagger + \bar{c}_{j\bar{1}}$, $f_j = \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger$, written in terms of the Hubbard operators $\bar{c}_{j\sigma}^\dagger = c_{j\sigma}^\dagger(1 - n_{j\bar{\sigma}})$ and $\bar{c}_{j\sigma} = (1 - n_{j\bar{\sigma}})c_{j\sigma}$ ($\sigma = 1, -1$), which form a subalgebra of the so-called double graded algebra $Spl(1,2)$ [4]. [A bar in a subindex means the negative of that number (e.g., $\bar{\sigma} = -\sigma$).] For spins on a lattice we fermionize the spins and reproduce the correct spin algebra with the following transformation:

$$\frac{S_j^+}{\sqrt{2}} = (\bar{c}_{j1}^\dagger K_j + K_j^\dagger \bar{c}_{j\bar{1}}), \qquad \frac{S_j^-}{\sqrt{2}} = (K_j^\dagger \bar{c}_{j1} + \bar{c}_{j\bar{1}}^\dagger K_j)\,,$$
$$S_j^z = \bar{n}_{j1} - \bar{n}_{j\bar{1}}\,,$$

whose inverse manifests the nonlocal character of the mapping

$$f_j^\dagger = \frac{1}{\sqrt{2}} \exp\left[i\pi \sum_{k<j} (S_k^z)^2\right] S_j^+\,,$$

$$f_j = \frac{1}{\sqrt{2}} \exp\left[-i\pi \sum_{k<j} (S_k^z)^2\right] S_j^-\,,$$

$$\bar{c}_{j1}^\dagger = S_j^z f_j^\dagger\,, \qquad \bar{c}_{j1} = f_j S_j^z\,,$$
$$\bar{c}_{j\bar{1}}^\dagger = -S_j^z f_j\,, \qquad \bar{c}_{j\bar{1}} = -f_j^\dagger S_j^z\,,$$

where the string operators $K_j = \exp[i\pi \sum_{k<j} \bar{n}_k] = \prod_{k<j}\prod_\sigma (1 - 2\bar{n}_{k\sigma})$, and the number operators $\bar{n}_k = \bar{n}_{k1} + \bar{n}_{k\bar{1}}$. These $f$ operators have the remarkable property that $\{f_j^\dagger, f_j\} = \{S_j^+, S_j^-\}$, which suggests an analogy

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between spins and "constrained" ($C$) fermions. To get an intuitive understanding of this transformation notice that the Hilbert space of a single spin $S = 1$ can be mapped onto the single-site Hilbert space of a two-flavor $C$ fermion [$S_z = -1, 0, 1$ maps onto $(\bar{n}_1, \bar{n}_{\bar{1}}) = (0,1), (0,0), (1,0)$; see Fig. 1 for the general case].

Half-odd integer spin chains have a qualitatively different excitation spectrum than integer spin chains. The Lieb, Schultz, Mattis, and Affleck theorem [5] establishes that the half-odd integer antiferromagnetic (AF) bilinear nearest-neighbor (NN) Heisenberg chain is gapless if the ground state is nondegenerate. The same model with integer spins is conjectured to display a Haldane gap [6]. To understand the origin of the Haldane gap we analyze the form of the 1D $S = 1$ $XXZ$ Hamiltonian using the above representation (an overall omitted constant $J > 0$ determines the energy scale)

$$H_{xxz} = \sum_j S_j^z S_{j+1}^z + \Delta(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) = \sum_j H_j^z + H_j^{xx}. \quad (3)$$

It is easy to show that the $C$ fermion version of this Hamiltonian is a ($S = \frac{1}{2}$) $t$-$J_z$ model [7] plus a particle nonconserving term which breaks the U(1) symmetry

$$H_{xxz} = \sum_j (\bar{n}_{j1} - \bar{n}_{j\bar{1}})(\bar{n}_{j+11} - \bar{n}_{j+1\bar{1}}) + \Delta \sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\sigma} + \bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\bar{\sigma}}^\dagger + \text{H.c.}). \quad (4)$$

In the isotropic $\Delta = 1$ limit, $H_{xxz}$ can be written as $H_{\rm Heis} = \sum_j (\Psi_j^\dagger \vec{\mathbf{S}} \Psi_j) \cdot (\Psi_{j+1}^\dagger \vec{\mathbf{S}} \Psi_{j+1})$, where $\vec{\mathbf{S}}$ is an irreducible matrix representation of $S = 1$ ($3 \times 3$ matrices) while $\Psi_j^\dagger = [\bar{n}_{j1}, (\bar{c}_{j1}^\dagger + \bar{c}_{j\bar{1}}^\dagger)K_j, \bar{n}_{j\bar{1}}]$.

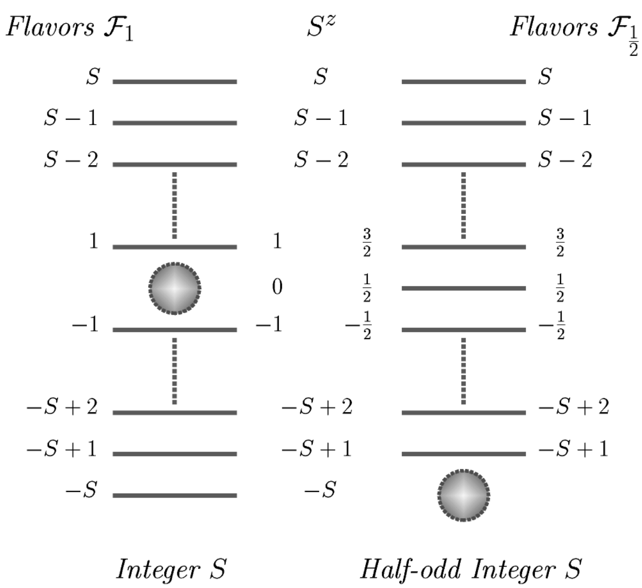


FIG. 1. Constrained fermion states per site for integer and half-odd integer spin $S$. In both cases there are $2S$ flavors and the corresponding $2S + 1$ values of $S^z$ are shown in the middle column. One degree of freedom is assigned to the fermion vacuum (circle) whose relative position depends upon the spin character.

The charge spectrum of the ($S = \frac{1}{2}$) $t$-$J_z$ model is gapless ($8|t| > J_z$) but the spin spectrum is gapped due to the explicitly broken SU(2) symmetry (Luther-Emery liquid) [7]. Therefore, the spectrum of the $S = 1$ Hamiltonian associated with the $t$-$J_z$ model with $t = -\Delta$ and $J_z = 4$ (which has only spin excitations) is gapless. Hence the term which explicitly breaks U(1) must be responsible for the opening of the Haldane gap. We can prove this by considering the perturbative effect that the interaction $\eta \sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\bar{\sigma}}^\dagger + \text{H.c.})$ has on the $t$-$J_z$ Hamiltonian. To linear order in $\eta$ ($>0$), $H_{\rm Heis}$ maps onto the ($S = \frac{1}{2}$) $XYZ$ model with $J_x = 2(\eta + 1)$, $J_y = -2(\eta - 1)$, and $J_z = -1$. From exact solution of this model [8], it is seen that the system is critical only when $\eta = 0$ while for $\eta \neq 0$ a gap to all excitations opens.

$S = 1$ *integrable models* [9].—To illustrate further the power of our spin-fermion mapping we now present *exact* solutions of 1D $S = 1$ models that have not been discovered by traditional techniques. These models correspond to the family of bilinear-biquadratic Hamiltonians,

$$H_1(\Delta) = \sum_j H_j^z + H_j^{xx} + \{H_j^z, H_j^{xx}\} = \sum_j H_j^z + \Delta \sum_\sigma \bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\sigma}, \quad (5)$$

that can be mapped onto a $t$-$J_z$ model, whose quantum phase diagram has recently been exactly solved [7].

Another well-studied class of bilinear-biquadratic SU(2) invariant Hamiltonians is [10]

$$H_2(\Delta) = \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \Delta(\mathbf{S}_j \cdot \mathbf{S}_{j+1})^2, \quad (6)$$

for $-1 \leq \Delta \leq 1$. The pure Heisenberg ($\Delta = 0$) and valence bond solid models ($\Delta = \frac{1}{3}$) belong to the Haldane gapped phase, which extends over the whole interval except at the boundaries $\Delta = \pm 1$ that are quantum critical points. The case $\Delta = -1$ is known to be Bethe ansatz soluble with a unique ground state and gapless. For $\Delta = 1$ we can map $H_2(1)$ onto the supersymmetric ($S = \frac{1}{2}$) $t$-$J$ Hamiltonian plus a NN repulsive interaction

$$H_2(1) = -\sum_{j\sigma} (\bar{c}_{j\sigma}^\dagger \bar{c}_{j+1\sigma} + \text{H.c.}) + 2 \sum_j \mathbf{s}_j \cdot \mathbf{s}_{j+1} + 2 \sum_j \left(1 - \bar{n}_j + \frac{3}{4} \bar{n}_j \bar{n}_{j+1}\right), \quad (7)$$

where $\mathbf{s}_j$ represents a $S = \frac{1}{2}$ operator. This model is Bethe ansatz soluble with a gapless phase [11] and is known as the Lai-Sutherland solution [12].

We now discuss the importance of our generalized JW transformation in unraveling hidden symmetries of an arbitrary spin Hamiltonian. In Eq. (6), for example, the $S = 1$ SU(2) symmetry is manifest. However, both the $S = \frac{1}{2}$ SU(2) and global U(1) gauge symmetries are hidden. On the other hand, in the transformed Hamiltonian, Eq. (7), these two symmetries are manifested explicitly through rotational invariance and charge conservation. The generators of these symmetries are related through the mapping already introduced. To illustrate this, we consider the U(1) symmetry case. Here the generator of the transformation is $Q = \sum_j \bar{n}_j$ which maps onto $Q = \sum_j (S_j^z)^2$ in the spin representation. The total group symmetry of the Hamiltonian is SU(3).

*Generalized transformation.*—A general transformation for arbitrary spin and spatial dimension is the following.

Half-odd integer spin $S$ ($\sigma \in \mathcal{F}_{1/2} = \{-S+1, \ldots, S\}$):

$$S_j^+ = \eta_{\bar{S}} \bar{c}^\dagger_{j\bar{S}+1} K_j + \sum_{\substack{\sigma \in \mathcal{F}_{1/2} \\ \sigma \neq S}} \eta_\sigma \bar{c}^\dagger_{j\sigma+1} \bar{c}_{j\sigma},$$

$$S_j^- = \eta_{\bar{S}} K_j^\dagger \bar{c}_{j\bar{S}+1} + \sum_{\substack{\sigma \in \mathcal{F}_{1/2} \\ \sigma \neq S}} \eta_\sigma \bar{c}^\dagger_{j\sigma} \bar{c}_{j\sigma+1},$$

$$S_j^z = -S + \sum_{\sigma \in \mathcal{F}_{1/2}} (S + \sigma) \bar{n}_{j\sigma},$$

$$\bar{c}^\dagger_{j\sigma} = K_j^\dagger L_\sigma^{1/2} (S_j^+)^{\sigma+S} \mathcal{P}_j^{1/2},$$

where $\mathcal{P}_j^{1/2} = \prod_{\tau \in \mathcal{F}_{1/2}} \frac{\tau - S_j^z}{\tau + S}, \qquad L_\sigma^{1/2} = \prod_{\tau=-S}^{\sigma-1} \eta_\tau^{-1}.$

Integer spin $S$ ($\sigma \in \mathcal{F}_1 = \{-S, \ldots, -1, 1, \ldots, S\}$):

$$S_j^+ = \eta_0 (\bar{c}^\dagger_{j1} K_j + K_j^\dagger \bar{c}_{j\bar{1}}) + \sum_{\substack{\sigma \in \mathcal{F}_1 \\ \sigma \neq -1, S}} \eta_\sigma \bar{c}^\dagger_{j\sigma+1} \bar{c}_{j\sigma},$$

$$S_j^- = \eta_0 (K_j^\dagger \bar{c}_{j1} + \bar{c}^\dagger_{j\bar{1}} K_j) + \sum_{\substack{\sigma \in \mathcal{F}_1 \\ \sigma \neq -1, S}} \eta_\sigma \bar{c}^\dagger_{j\sigma} \bar{c}_{j\sigma+1},$$

$$S_j^z = \sum_{\sigma \in \mathcal{F}_1} \sigma \bar{n}_{j\sigma},$$

$$\bar{c}^\dagger_{j\sigma} = K_j^\dagger L_\sigma^1 \begin{cases} (S_j^+)^\sigma \mathcal{P}_j^1 & \text{if } \sigma > 0, \\ (S_j^-)^\sigma \mathcal{P}_j^1 & \text{if } \sigma < 0, \end{cases}$$

where $\mathcal{P}_j^1 = \prod_{\tau \in \mathcal{F}_1} \frac{\tau - S_j^z}{\tau}, \qquad L_\sigma^1 = \prod_{\tau=0}^{|\sigma|-1} \eta_\tau^{-1},$

and $\eta_\sigma = \sqrt{(S-\sigma)(S+\sigma+1)}$ (see Fig. 1).

The total number of flavors is $N_f = 2S$, and the $S = \frac{1}{2}$ case simply reduces to the traditional JW transformation. Since these mappings are exact they preserve the invariant Casimir operator $\mathbf{S}_j^2 = S(S+1)$. The generalized $C$ fields

$$\bar{c}^\dagger_{j\sigma} = c^\dagger_{j\sigma} \prod_{\tau \in \mathcal{F}_\alpha} (1 - n_{j\tau}), \qquad \bar{c}_{j\sigma} = \prod_{\tau \in \mathcal{F}_\alpha} (1 - n_{j\tau}) c_{j\sigma} \tag{8}$$

form a subalgebra of the generalized Hubbard double graded algebra, where the "unconstrained" operators $c^\dagger_{j\sigma}, c_{j\sigma}$ satisfy the standard fermion anticommutation relations ($\alpha = \frac{1}{2}, 1$ depending upon the spin character of the representation). These generalized $C$ operators (only single occupancy is allowed) anticommute for different sites

$$\{\bar{c}_{j\sigma}, \bar{c}_{k\sigma'}\} = \{\bar{c}^\dagger_{j\sigma}, \bar{c}^\dagger_{k\sigma'}\} = 0,$$
$$\{\bar{c}_{j\sigma}, \bar{c}^\dagger_{k\sigma'}\} = \delta_{jk} \begin{cases} \prod_{\substack{\tau \in \mathcal{F}_\alpha \\ \tau \neq \sigma}} (1 - \bar{n}_{j\tau}) & \text{if } \sigma = \sigma', \\ \bar{c}^\dagger_{j\sigma'} \bar{c}_{j\sigma} & \text{if } \sigma \neq \sigma', \end{cases} \tag{9}$$

and their number operators satisfy $\bar{n}_{j\sigma} \bar{n}_{j\sigma'} = \delta_{\sigma\sigma'} \bar{n}_{j\sigma}$.

The string operators $K_j$ introduce nonlinear and nonlocal interactions between the $C$ fermions. For 1D lattices ($K_j = K_j^\dagger$, $[K_i, K_j] = 0$) they are the so-called kink operators $K_j = \exp[i\pi \sum_{k<j} \bar{n}_k]$, while for 2D [13]

$$K_{\mathbf{j}} = \exp\left[ i \sum_{\mathbf{k}} a(\mathbf{k}, \mathbf{j}) \bar{n}_{\mathbf{k}} \right], \tag{10}$$

$$\text{with } \bar{n}_{\mathbf{k}} = \sum_{\sigma \in \mathcal{F}_\alpha} \bar{n}_{\mathbf{k}\sigma} = 1 - \mathcal{P}_{\mathbf{k}}^\alpha. \tag{11}$$

Here $a(\mathbf{k}, \mathbf{j})$ is the angle between the spatial vector $\mathbf{k} - \mathbf{j}$ and a fixed direction on the lattice, and $a(\mathbf{j}, \mathbf{j})$ is defined to be zero. We comment that the 1D kink operators constitute a particular case of Eq. (10) with $a(k, j) = \pi$ when $k < j$ and equals zero otherwise. For $d > 2$, the string operators generalize [9] along the lines introduced in Ref. [14].

There is always the freedom to perform rotations in spin space to get equivalent representations to the one presented above. However, for bilinear isotropic NN Heisenberg [spin SU(2) rotationally invariant] Hamiltonians in the large-$S$ limit there is a fundamental difference between effective integer and half-odd integer spin cases. In the latter case a new local U(1) gauge symmetry emerges that is explicitly broken in the integer case. For 1D lattices, this is precisely what distinguishes Haldane gap systems [6] from half-odd integer spin chains that are critical.

We mention that other fermionic representations are feasible. In particular, for half-odd integer cases where $2S + 1 = \sum_{i=0}^{\bar{N}_f} \binom{\bar{N}_f}{i} = 2^{\bar{N}_f}$ (e.g., $S = \frac{3}{2}$ with $\bar{N}_f = 2$) a simple transformation in terms of canonical fermions is possible [9]. For these mappings the string operators must be modified to take into account the double occupancy of a site. In this way the Hubbard model can be mapped onto a $S = \frac{3}{2}$ spin Hamiltonian [9].

*2D Lattices and spin-anyon mapping.*—The generalization of these transformations to higher dimensions gives new exact mappings between spin theories and $C$ fermion systems in the presence of gauge fields. To illustrate this we write the $S = 1$ Hamiltonian $H_2(1)$ in the fermion representation for $d = 2$

$$\begin{aligned} H_2(1) = &-\sum_{\mathbf{j}\sigma,\nu} (\bar{c}^\dagger_{\mathbf{j}+\mathbf{e}_\nu\sigma} e^{-iA_\nu(\mathbf{j})} \bar{c}_{\mathbf{j}\sigma} + \text{H.c.}) \\ &+ 2 \sum_{\mathbf{j},\nu} \mathbf{s}_{\mathbf{j}} \cdot \mathbf{s}_{\mathbf{j}+\mathbf{e}_\nu} \\ &+ \sum_{\mathbf{j},\nu} \left( 2 - (\bar{n}_{\mathbf{j}} + \bar{n}_{\mathbf{j}+\mathbf{e}_\nu}) + \frac{3}{2} \bar{n}_{\mathbf{j}} \bar{n}_{\mathbf{j}+\mathbf{e}_\nu} \right), \end{aligned} \tag{12}$$

and $A_\nu(\mathbf{j}) = \sum_{\mathbf{k}} [a(\mathbf{k}, \mathbf{j}) - a(\mathbf{k}, \mathbf{j} + \mathbf{e}_\nu)]\bar{n}_{\mathbf{k}}$, where $\mathbf{e}_\nu$ ($\nu = 1, 2$) are basis vectors of the Bravais lattice connecting NN and $\mathbf{j}$'s represent sites of the corresponding 2D lattice. We note that the field $A_\nu(\mathbf{j})$ is associated with the change in particle statistics. It is well known [4,13] that the same transmutation of particle statistics can be achieved via a path-integral formulation for $H_2(1)$ where an Abelian lattice Chern-Simons term is included. In this formulation a constraint (Gauss's law) requiring that the gauge flux through a plaquette $\mathbf{j}$ be proportional to the total fermion density on the site, $\bar{n}_{\mathbf{j}}$, is enforced. This suggests that our spin-fermion mapping can be generalized to a spin-anyon transformation with a hard-core condition for the anyon fields [9]. In fact, one can formally take our generalized JW transformation and replace the string operators $K_{\mathbf{j}}$ by the statistical operators $K_{\mathbf{j}}(\theta) = \exp[i\theta \sum_{\mathbf{k}} a(\mathbf{k}, \mathbf{j})\bar{n}_{\mathbf{k}}]$ with $0 \leq \theta \leq 1$. With this choice, the $\bar{c}$ operators satisfy equal-time anyon commutation relations [$\theta = 1(0)$ corresponds to $C$ fermions (bosons)] [9]. Similar ideas apply for 1D lattices.

One immediately sees the relevance of these transformations for the theories of magnetism and high-temperature superconductivity: A class of $S = 1$ Hamiltonians that can be mapped onto a lattice-gauge (Chern-Simons) $S = \frac{1}{2}$ $t$-$J$ theory and vice versa, for example, a $S = \frac{1}{2}$ $t$-$J$ model,

$$H_{t-J} = -t \sum_{\mathbf{j}\sigma,\nu} (\bar{c}^\dagger_{\mathbf{j}\sigma} \bar{c}_{\mathbf{j}+\mathbf{e}_\nu\sigma} + \text{H.c.}) + J \sum_{\mathbf{j},\nu} \mathbf{s}_{\mathbf{j}} \cdot \mathbf{s}_{\mathbf{j}+\mathbf{e}_\nu} - \mu \sum_{\mathbf{j}} \bar{n}_{\mathbf{j}} , \qquad (13)$$

can be exactly mapped onto a lattice-gauge bilinear-biquadratic $S = 1$ theory

$$H_{t-J} = -\mu \sum_{\mathbf{j}} (S^z_{\mathbf{j}})^2 + \frac{J}{8} \sum_{\mathbf{j},\nu} \left[ H^z_{\mathbf{j}\nu} - \frac{4t}{J} S^+_{\mathbf{j}} e^{iA_\nu(\mathbf{j})} S^-_{\mathbf{j}+\mathbf{e}_\nu} - \frac{4t}{J} \{H^z_{\mathbf{j}\nu}, S^+_{\mathbf{j}} e^{iA_\nu(\mathbf{j})} S^-_{\mathbf{j}+\mathbf{e}_\nu}\} + (S^+_{\mathbf{j}} S^-_{\mathbf{j}+\mathbf{e}_\nu})^2 + \text{H.c.} \right]. \qquad (14)$$

By means of a semiclassical approximation it has been shown [15] that the ground state of $H_2(1)$ is on the boundary between AF ($\Delta < 1$) and orthogonal nematic (nonuniform, $\Delta > 1$) phases [10,15]. These two states are the result of the competition between the quadratic and quartic spin-exchange interactions. In terms of the equivalent $t$-$J$ gauge theory this translates into a competition between antiferromagnetism and delocalization. Qualitatively, the string path of the particle moving in an AF background gives rise to a linear confining potential since the number of frustrated magnetic links is proportional to the length of the path. This observation suggests that the inhomogeneous phases observed in the "striped" high-$T_c$ compounds can be driven by the competition between magnetism and delocalization.

*Summary.*—We introduced a general spin-fermion mapping for arbitrary spin $S$ and spatial dimension that naturally generalizes the Jordan-Wigner transformation for $S = \frac{1}{2}$. Mathematically, we established a one-to-one mapping of elements of a Lie algebra onto elements of a fermionic algebra with a hard-core constraint. Several generalizations, like a spin-anyon mapping, and important consequences result from these transformations [9]. For instance, the well-known transformation between $S = \frac{1}{2}$ and hard-core bosons in any dimension [16] is a particular case of our general mappings. Incidentally, we note that there are extremely powerful numerical techniques (cluster algorithms [17]) to study quantum spin systems, and our mapping allows one to extend these methods to study the equivalent fermionic problems.

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[1] D. C. Mattis, *The Theory of Magnetism I* (Springer, Berlin, 1988).

[2] P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).

[3] For a $d$-dimensional lattice with $N$ sites, the operator $S^\mu_j$ is defined in terms of a Kronecker product $\otimes$ as

$$S^\mu_j = \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \underbrace{S^\mu}_{j\text{th}} \otimes \cdots \otimes \mathbb{1} ,$$

where $\mathbb{1}$ is the $(2S + 1) \times (2S + 1)$ unit matrix and $S^\mu$ is a spin-$S$ operator. Thus $S^\mu_j$ admits a matrix representation of dimension $(2S + 1)^N \times (2S + 1)^N$.

[4] A. M. Tsvelik, *Quantum Field Theory in Condensed Matter Physics* (Cambridge University Press, Cambridge, 1995).

[5] E. H. Lieb, T. D. Schultz, and D. C. Mattis, Ann. Phys. **16**, 407 (1961); I. Affleck and E. H. Lieb, Lett. Math. Phys. **12**, 57 (1986).

[6] F. D. M. Haldane, Phys. Lett. **93A**, 464 (1983).

[7] C. D. Batista and G. Ortiz, Phys. Rev. Lett. **85**, 4755 (2000).

[8] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1990).

[9] More examples and details will be published elsewhere.

[10] G. Fáth and J. Sólyom, Phys. Rev. B **51**, 3620 (1995).

[11] P. Schlottmann, Phys. Rev. B **36**, 5177 (1987).

[12] C. K. Lai, J. Math. Phys. (N.Y.) **15**, 1675 (1974); B. Sutherland, Phys. Rev. B **12**, 3795 (1975).

[13] E. Fradkin, Phys. Rev. Lett. **63**, 322 (1989); D. Eliezer and G. W. Semenoff, Phys. Lett. B **286**, 118 (1992).

[14] L. Huerta and J. Zanelli, Phys. Rev. Lett. **71**, 3622 (1993).

[15] N. Papanicolaou, Nucl. Phys. **B305**, 367 (1988).

[16] T. Matsubara and H. Matsuda, Prog. Theor. Phys. **16**, 569 (1956).

[17] J. E. Gubernatis and N. Kawashima, in *Monte Carlo and Molecular Dynamics of Condensed Matter Systems,* edited by K. Binder and G. Ciccotti (Società Italiana de Fisica, Bologna, 1995).