

Phase Transitions in Liquid ^3He

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The phase transitions of liquid ^3He are described by truncations of an exact nonperturbative renormalization group equation. The location of the first-order transition lines and the jump in the order parameter are computed quantitatively. At the triple point we find indications of partially universal behavior. We suggest experiments that could help to determine the effective interactions between fermion pairs.

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The low temperature phase transitions and the superfluid phases of liquid ^3He [1] can be described [2] by a field theory for complex 3×3 -matrices A representing fermion pairs. An approximation based on quartic polynomials in A , including the renormalization group running of the coupling constants [3], can account for the rough phase structure but not for details of the transitions. We will be concerned with the transitions from the normal liquid to superfluidity in vanishing external magnetic field. In a mean-field treatment those transitions are of second order, whereas early renormalization-group calculations [3] found hints for them to be fluctuation-induced first-order transitions. The Bardeen-Cooper-Schrieffer weak-coupling theory estimates the critical region where fluctuations invalidate the mean-field approximation to extend over a temperature interval $\Delta T \approx 10^{-8}$ K. Based on experiments on zero sound absorption [4] it has been conjectured, however, that the critical region might be up to a thousand times larger [3]. The theoretical answers therefore hinge on two problems: first, the “microphysical” effective interactions at the scale of an effective bosonic theory for fermion pairs (we assume $\Lambda^{-1} = 120 \text{ \AA}$) are poorly known and, second, the transition from the microphysical interactions to the macrophysics of the phase transitions (i.e., the thermodynamic potentials) is very complex. Only if the second problem can be solved can the experimental observations of the macrophysics (phase structure, jump in the order parameter, etc.) be used to constrain the microphysical interactions. In this Letter we propose a quantitatively reliable mapping of the microphysics onto the properties of the free energy density. This is effected by means of extended truncations to an exact nonperturbative renormalization group equation [5], in contrast to fourth-order polynomial approximations used earlier [6,7]. A closer examination of the polynomial approximation to sixth order shows that it does not lead to a convergent and reliable description of the phase diagram.

The different phases of ^3He are characterized [8] by expectation values $\langle A \rangle$ (order parameters) in different directions: the Balian-Werthamer state, $A_1 = 1$, the Anderson-Brinkman-Morel state, $A_2 = \sqrt{\frac{3}{8}}(\lambda_7 + i\lambda_6 + \lambda_4 - i\lambda_5)$, and the planar state, $A_3 = \sqrt{\frac{3}{2}}i\lambda_2$

represent the phases of the mean-field diagram. The λ_i are the Gell-Mann matrices. Additionally, we examine the direction $A_4 = \sqrt{\frac{3}{4}}(\lambda_3 - i\lambda_1)$. Here, $\langle A \rangle$ corresponds to the minimum of the effective potential $U(A)$ for spatially homogeneous fields. Our main aim is therefore the computation of $U(A)$ for a given microscopic effective action (at the scale Λ) for which we assume the conventional form [2]

$$\Gamma_\Lambda = \int d^3x \left\{ \text{Tr} \vec{\nabla} A^\dagger \vec{\nabla} A + m\rho + \sum_{i=0}^4 b_i I_i + aK_D \right\} \quad (1)$$

with $\rho = \text{Tr} A^\dagger A$ and I_i the invariants to fourth order allowed by the symmetry group $G = \text{SO}(3) \times \text{SO}(3) \times U(1)$: $I_0 = \rho^2$, $I_1 = |\text{Tr} A^T A|^2$, $I_2 = \text{Tr}(A^T A)(A^\dagger A)^*$, $I_3 = \text{Tr}(A^\dagger A)^2$, and $I_4 = \text{Tr}(A^\dagger A)(A^\dagger A)^*$. We include the dipole interaction

$$K_D = \text{Tr} A^\dagger \text{Tr} A + \text{Tr}(A^* A) \quad (2)$$

as a symmetry breaking perturbation arising from the spin-orbit coupling of the atoms in a Cooper pair and neglect the strain gradient terms [2]. All quantities are in units of a characteristic critical temperature $T_c = 2.6$ mK, and $m(T) = m(T_c) + \frac{T-T_c}{T_c} + \dots$ reflects the temperature dependence. In the paramagnon theory the quartic couplings b_i are given by [3]

$$\begin{aligned} b_0 &= c(2 + 0.2\delta), & b_1 &= -c(1 + 0.1\delta), \\ b_2 &= c(2 - 0.05\delta), & b_3 &= c(2 - 0.55\delta), \\ b_4 &= -c(2 + 0.7\delta) \end{aligned} \quad (3)$$

with $c = 0.001$ and δ parametrizing the pressure dependence. We study a pressure regime of $-2 < \delta < 1.8$. For $\delta > 1.9$ the microscopic potential becomes unbounded from below. This will be compared to a calculation where the couplings are 10 times as strong.

Our approach is based on the effective average action $\Gamma_k[A(x)]$ which interpolates between the microphysics ($k = \Lambda$) and the macrophysics ($k = 0$) by means of an exact flow equation [5]. We truncate the most general functional dependence on $A(x)$ in lowest order of a derivative expansion with nonzero anomalous dimension [9],

$$\Gamma_k = \int d^3x \{Z_k \text{Tr} \vec{\nabla} A^\dagger \vec{\nabla} A + U_k(A) + a_k K_D\}. \quad (4)$$

Here $U_k(A)$ is a function of the 18 real fields φ_y in A . It respects the symmetry group G . We can compute the effective potential $U(A) \equiv U_0(A)$ by a numerical integration of the flow equation [5]

$$\begin{aligned} \partial_t U_k(A) &= \frac{1}{2} \int \frac{d^3q}{(2\pi)^3} \partial_t R_k \\ &\times \text{Tr}[(Z_k q^2 + R_k) + M_k^2(A)]^{-1}. \end{aligned} \quad (5)$$

Here, $t = \ln k$. Our choice of the infrared cutoff,

$$R_k(q) = Z_k q^2 (e^{q^2/k^2} - 1)^{-1}, \quad (6)$$

makes the flow equation ultraviolet and infrared finite. We need the mass matrix of the second derivatives of U , $M_{y,z}^2 = \frac{\partial^2 U}{\partial \varphi_y \partial \varphi_z}$. The flow equation for the wave function renormalization Z_k can be found in [9]. We evaluate it at the minimum of the potential.

In a first investigation we used a sixth-order polynomial approximation for U_k and searched for minima in all the 15 directions with different residual symmetries given in [8]. In this scheme we found a phase transition in the direction A_4 for large δ . A detailed investigation of a general form of the potential shows, however, that this transition is an artifact of the polynomial approximation. We conclude that polynomial approximations do not allow reliable statements about the phase diagram. Instead, we deal with an arbitrary dependence of U_k on the field in the four given directions, $A = \xi A_n$, for which we write $U_k(\xi A_n) = V_k^{(n)}(\rho)$ with $\rho = 3\xi^2$. We assume that the directions n include the absolute minimum of U_k . We get the flow equation for $V_k^{(n)}$ by evaluating (5) at points on the line ξA_n . To do so, we need to know the second derivatives of U_k with respect to all fields, however. That is why we include in our truncation four coupling functions $b_i^{(n)}(\rho)$ multiplying all fourth-order invariants other than ρ^2 :

$$\begin{aligned} U(\xi A_n + \epsilon) &= \left\{ V^{(n)}(\rho) + \sum_{i=1}^4 b_i^{(n)}(\rho) \right. \\ &\left. \times (I_i - f_i^{(n)} \rho^2) \right\} \Big|_{\xi A_n + \epsilon}. \end{aligned} \quad (7)$$

The numbers $f_i^{(n)}$ are chosen such that the invariants $I_i - f_i^{(n)} \rho^2$ do not contribute on the line ξA_n . Here ϵ denotes fields orthogonal to ξA_n and we expect that this truncation gives a good approximation to the true potential in a neighborhood of the line ξA_n .

The flow equations of the coupling functions $b_i^{(n)}(\rho)$ are derived by forming appropriate linear combinations of second and fourth derivatives of (5) with respect to certain fields. We have computed these flow equations for the four field directions A_n . The resulting equations are very long, altogether about 4000 lines. They all derive, however, from compact expressions, namely, field derivatives of (5), which can be evaluated by a computer. We consider this

structural simplicity to be one of the major strengths of our formalism.

We have integrated the resulting equations numerically by laying the potentials and coupling functions on grids of ten points each. In a first run, we detected truncation errors in the quantities $\frac{\partial V^{(n)}}{\partial \rho} \Big|_{A=0}$. They originate in an insufficient treatment of the symmetry constraints. We fight this problem in two steps. First, we infer the first potential derivatives at the origin from the term $m\rho$ and integrate a separate flow equation for m . Second, for the higher orders we employ for small $|t|$ a mixed truncation of the form

$$\begin{aligned} U(\xi A_n + \epsilon) &= \left\{ m\rho + \sum_{i=0}^4 b_i I_i + \sum_{j=0}^{10} c_j J_j + \rho^4 w^{(n)}(\rho) \right. \\ &\left. + \sum_{i=1}^4 \rho^2 B_i^{(n)}(\rho) (I_i - f_i^{(n)} \rho^2) \right\} \Big|_{\xi A_n + \epsilon}. \end{aligned} \quad (8)$$

Here J_j are the 11 invariants of sixth order, and $w^{(n)}$ and $B_i^{(n)}$ describe the deviation from a sixth-order polynomial approximation. The symmetry is manifestly respected up to sixth order. We have estimated the accuracy of the sixth-order polynomial approximation by calculating $\partial_t U(\xi A_n)$ to fourth and sixth order in ξ in two ways: from the flow of the couplings in the polynomial part of truncation (8) and directly by Eq. (5). The discrepancy stays smaller than 20% as long as the flow parameter $|t|$ is not too big, $t > -5$. It grows to a few hundred percent, however, until the phase transition occurs at $t \approx -8$. This is due to terms of eighth and higher order that are truncated in the flow equations for the couplings b_i and c_j in (8). Again we observe, now on a more technical level, that a polynomial truncation of the free energy to realistic order does not permit any conclusive statements about the phase diagram. As soon as the errors exceed 20%, we switch to the truncation (7). To summarize, the computation of the thermodynamic potential U_0 for a microscopic action Γ_Λ contains two possible sources of errors. First, it has been argued that the errors from the truncation (4) are governed by the size of the anomalous dimension [9] which is only a few percent in our case. Second, inaccuracies arise from the numerical solution of Eq. (5). We estimate the error in the quantities $\partial_t V^{(n)}$ at the potential minima by comparing truncation (7) with an alternative truncation in which we extract the orthogonal masses from the polynomial truncation in the origin. We find this error to be no bigger than a few percent. Also the errors induced by the inaccuracies of $\frac{\partial V^{(n)}}{\partial \rho} \Big|_{A=0}$ are of this order of magnitude. Finally, we find no indication that our results depend on the precise choice of the cutoff function R_k in any important way. Our approximations should therefore allow us to compute the phase structure reliably.

We now study the case of large couplings, $c = 0.01$. We neglect the spin-orbit coupling, which is justified in

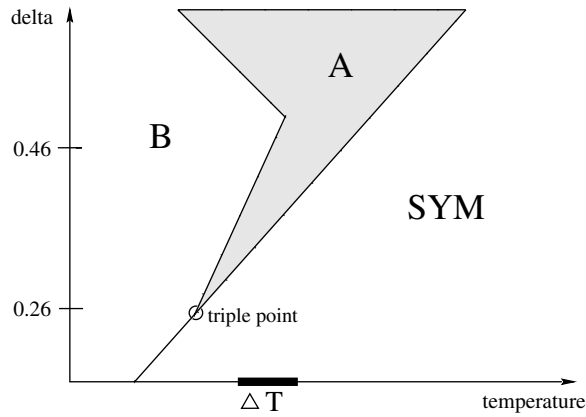


FIG. 1. Phase diagram of the effective theory for liquid ³He with strong coupling ($c = 0.01$). The temperature scale is set by $\Delta T = 10^{-6}$ K. The same picture is found for $c = 0.001$, neglecting the spin-orbit interaction. Then $\Delta T = 10^{-8}$ K.

this case, the other masses being much bigger. By computing the temperature dependence of the renormalization of the fourth-order couplings, we find a width of the critical region of about 10^{-6} K, in good agreement with the Ginzburg criterion. The resulting phase diagram is shown in Fig. 1. In a pressure regime of $0.26 < \delta < 0.46$ a beak of A-phase between the symmetric and the B-phase is stabilized by the fluctuations of the order parameter. In the middle of this pressure interval the temperature width of this wedge is about 10^{-6} K. Towards the upper end it rises, however, to 10^{-5} K at $\delta = 0.41$. All transitions are first order.

Figure 2 shows the pressure dependence of the correlation length, given by the inverse of the renormalized mass at the potential minimum, and the discontinuity of the NMR frequency shift at the phase transition. The latter quantity is related to the jump in the order parameter squared $\Delta\rho$ by

$$(\Delta\nu)^2 \approx 6c \times 10^{10} \text{ Hz} \times \Delta\rho \quad (9)$$

which follows from the mean-field temperature dependence of $\Delta\nu$ given in [2]. We observe a substantial

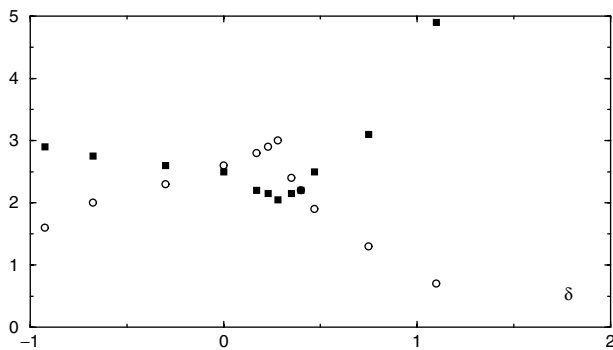


FIG. 2. Pressure dependence of the correlation length at the phase transition (in μm , circles) and the discontinuity in the NMR frequency shift (in kHz, squares); $c = 0.01$.

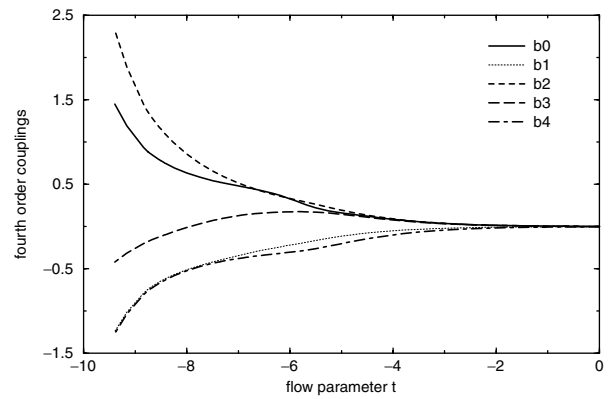


FIG. 3. Flow of the dimensionless renormalized fourth-order couplings near the triple point ($c = 0.001$, $\delta = 0.27$, $m \approx 0.004$, no spin-orbit coupling).

increase of the correlation length as we approach the triple point $\delta = 0.26$. This observation nourishes speculations about a fixed point in the vicinity of this point in parameter space which would give rise to universal behavior of the system. Accurate predictions could then be made despite the uncertainty of our knowledge of the microscopic theory. Also an examination of the flow of the couplings at the origin looks interesting near the triple point. We observe strong renormalization of the renormalized fourth-order couplings b_{i_r} (Fig. 3).

For example, $b_{4_A}/b_{4_0} \approx 8$ (the b_{i_k} are the unrenormalized, dimensionful couplings at scale k), whereas analogous ratios away from the triple point, say, at $\delta = 0$, grow no larger than 1.5 at the phase transition. Even though we do not find a true fixed point behavior the ratios of certain couplings seem to tend to constant values, such as $b_{1_r}/b_{4_r} = 1$, corresponding to a partial fixed point.

The qualitative features for smaller couplings $c = 0.001$ look similar, with correspondingly smaller $\Delta T = 10^{-8}$ K. In this case, however, the dipole term

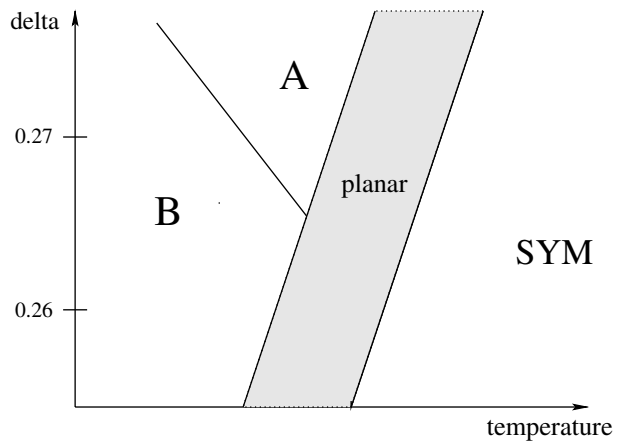


FIG. 4. Phase diagram of the effective theory for liquid ³He in the parameter range suggested by the paramagnon model close to the transition line. The temperature width of the planar phase is about 10^{-9} K.

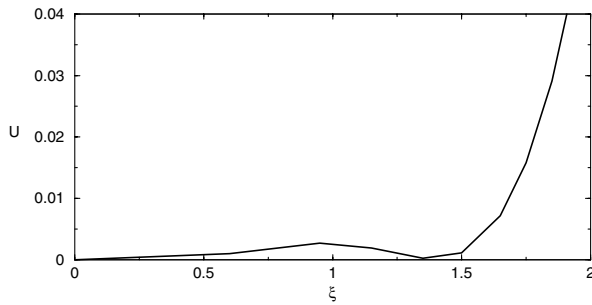


FIG. 5. Potential U_{k_f} in the planar direction at the phase transition. We show the dependence on $\xi = \sqrt{\frac{p}{3}}$.

should be included ($a_\Lambda = 2 \times 10^{-6}$) which we do next. Figure 4 shows the phase diagram with first-order transition lines, similar to [2]. The width of the stripe of planar phase stabilized by the dipole interaction is about 2×10^{-9} K as expected from mean-field considerations. Farther away from the transition to the symmetric phase the effect of the dipole term can be neglected, giving rise to the structure depicted in Fig. 1.

By tuning the temperature to the transition from the symmetric to the planar phase, we find a potential U_{k_f} (Fig. 5) which shows clearly that the transition is of first order. (Fluctuations with momenta smaller than k_f make the potential convex without much influence on the equation of state [10].)

From Fig. 5 we can easily infer the order parameter jump: $\Delta\rho \approx 8 \times 10^{-5}$. This turns out to be almost independent of the pressure in the regime examined by us. It corresponds to a discontinuity in the NMR frequency shift at the transition of about 50 Hz. It was suggested in [2] that this should be experimentally observable.

Let us finally ask how experiments may constrain the size of the couplings. We have seen that large couplings lead to a widening of the critical region ΔT . It should be possible to exploit this fact by measuring the width of the critical region. Varying the temperature at a pressure right under the triple point, one should be able to observe two distinct phase transitions. The width of the intermediate phase relates to the coupling strength via the Ginzburg criterion. By using the described experiment one could also measure the pressure corresponding to the values $\delta = 0.26$ and $\delta = 0.46$. An approximate form of the relation $p(\delta)$ could then be obtained by linearization. Strictly speaking, these arguments hold only if the dipole term can be neglected. However, the temperature range for the planar phase, 10^{-9} K, is so narrow that it would probably not even show up in experiments with realistic temperature resolution.

Also, the order parameter jump depends strongly on the coupling strength. With the larger couplings $c = 0.01$ we

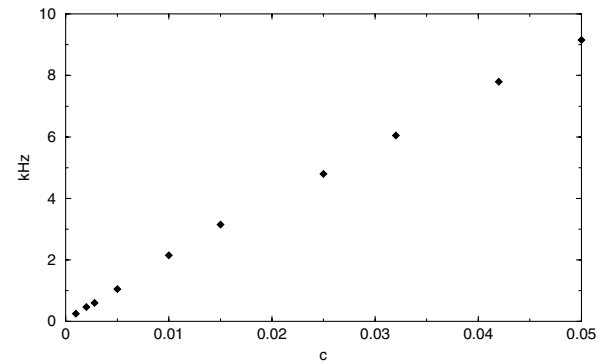


FIG. 6. Dependence of the discontinuity of the frequency shift (in kHz) on the coupling strength c .

find $\Delta\rho \approx 0.01$ for the jump into the B -phase at the triple point. This leads to a discontinuity in the frequency shift of about 2000 Hz as opposed to, e.g., 370 Hz for $c = 0.002$. We think that it would be even easier to infer the size of the critical region from measuring this order parameter jump. At least, this way it should be possible to get upper bounds for the coupling strength that are rather close to the range discussed in this article. To facilitate the interpretation of such NMR experiments, we calculated the jump of $\Delta\nu$ for various other values of the coupling strength c at $\delta = 0.15$. We show the result in Fig. 6.

In conclusion, our quantitative description of the phase diagram of ^3He should permit one to extract reliable information about the system's microphysical interactions from future experiments.

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