Nonperturbative Lorentzian Path Integral for Gravity

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We construct a well-defined regularized path integral for Lorentzian quantum gravity in terms of dynamically triangulated causal space-times. Each Lorentzian geometry and its action have a unique Wick rotation to the Euclidean sector. All space-time histories possess a distinguished notion of a discrete proper time and, for finite lattice volume, the associated transfer matrix is self-adjoint, bounded, and strictly positive. The degenerate geometric phases found in dynamically triangulated Euclidean gravity are not present.

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There seems to be a broad consensus that a correct nonperturbative treatment of quantum gravity should involve in an essential way the full "space of all metrics" (as opposed to linearized perturbations around flat space) and the diffeomorphism group, i.e., the invariance group of the classical gravitational action,

$$S[g_{\mu\nu}] = \frac{k}{2} \int d^d x \sqrt{-\det g}\,(R - 2\Lambda), \qquad d = 4, \tag{1}$$

with $k^{-1} = 8\pi G_N$ and the cosmological constant Λ.

One approach that does not rely on the existence of supersymmetry tries to define the theory by means of a nonperturbative path integral. The aim is not to evaluate this in a stationary-phase approximation, but as a genuine "sum over all geometries." Since even the pure gravity theory in spite of its large invariance group possesses local field degrees of freedom, such a sum must be regularized to make it meaningful. The two most popular approaches, quantum Regge calculus and dynamical triangulations (see [1] for recent reviews), both employ simplicial discretizations of space-time, on which then the behavior of metric and matter fields is studied. One drawback of these (mainly numerical) investigations is that so far they have been conducted only for *Euclidean* space-time metrics $g^E_{\mu\nu}$ instead of the physical, Lorentzian metrics. This is motivated by the analogy with nonperturbative Euclidean (lattice) field theories on a fixed, flat background, whose results can under suitable conditions be "Wick rotated" to their Minkowskian counterparts. The amplitudes $\exp iS$ are substituted in the Euclidean path integral by the real weights $\exp(-S^E)$. *Real* weight factors are mandatory for the convergence of the state sums.

Unfortunately, it is not clear how to relate path integrals over Euclidean geometries to those over Lorentzian ones. For general metric configurations, there is no preferred notion of "time" and hence no obvious prescription of how to Wick rotate.

One might worry that with a discretization of space-time the diffeomorphism invariance of the continuum theory is irretrievably lost. However, the example of two-dimensional Euclidean gravity shows that this is not necessarily so, since there one obtains agreement between the scaling limit of the discrete theory and continuum Liouville quantum gravity.

In the approach of *dynamical triangulations,* the sum over all Euclidean metrics modulo diffeomorphisms is approximated by a sum over all equilateral triangulations of a given topological manifold. (It should be emphasized that this is an intrinsically *quantum* formulation, and not directly suitable as an approximation scheme for classical gravity.) Since different triangulations correspond to different geometries, this regularization has no residual gauge invariance. This makes it an appealing method for investigating quantum gravity theories, and extensive numerical simulations have been conducted in dimensions 2, 3, and 4. Alas, in $d = 3, 4$ an interesting continuum limit has not been found. This seems to be related to the dominance of degenerate geometries. At small bare gravitational coupling G_N, the dominant configurations are branched polymers (or "stacked spheres") with Hausdorff dimension $d_H = 2$, whereas at large G_N the geometries "condense" around one or several singular links or vertices with a very high coordination number, resulting in a very large d_H. These extreme geometric phases are separated by a first-order phase transition. Another unsatisfactory aspect of the Euclidean model is our inability to rotate back to Lorentzian space-time.

In order to tackle these problems, two of us have recently constructed a Lorentzian version of the dynamically triangulated gravitational path integral in *two* dimensions [2]. The individual geometries are glued together from Lorentzian triangles in a way that satisfies certain causality requirements. The model is exactly soluble and its associated continuum theory lies in a new universality class of 2D gravity models distinct from the usual Euclidean Liouville gravity. One central lesson from this example is that the causality conditions imposed on the Lorentzian model act as a "regulator" for the geometry. Most importantly, they suppress changes in the spatial topology, that

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is, a branching of baby universes. As a result, the effective quantum geometry in the Lorentzian case is smoother and in some senses better behaved: (a) In spite of large fluctuations of the geometry, its Hausdorff dimension has the canonical value $d_H = 2$, unlike what happens in Euclidean gravity, which has a fractal dimension $d_H = 4$; (b) in spite of a strong interaction between matter and gravity when the system is coupled to Ising spins, the combined system remains consistent even beyond the $c = 1$ barrier, unlike what happens in Euclidean gravity [3,4].

Motivated by these results, we have constructed a discretized Lorentzian path integral for gravity in three and four space-time dimensions. Unlike what happens in two dimensions, the action is no longer trivial, and the Wick-rotation problem must be solved. We have succeeded in constructing a model with the following properties: (i) Lorentzian space-time geometries are obtained by causally gluing sets of d-dimensional Lorentzian building blocks; (ii) all histories have a preferred discrete notion of proper time t, counting the number of evolution steps of a transfer matrix between adjacent spatial slices; (iii) for a fixed space-time volume N_d, both the Euclidean and the Lorentzian discretized gravity actions are bounded from above and below; (iv) the number of possible triangulations is exponentially bounded as a function of the space-time lattice volume; (v) each Lorentzian discrete geometry can be Wick rotated to a Euclidean one, defined on the same (topological) triangulation; (vi) a "Wick rotation" is achieved by an analytical continuation of the discretized action in the dimensionless ratio $\alpha = -l^2_{\text{time}}/l^2_{\text{space}}$ of the squared time- and spacelike link length; (vii) for finite lattice volume, the discrete transfer matrix is a self-adjoint, bounded operator which is strictly positive; and (viii) the extreme phases of degenerate geometries found in the Euclidean models cannot be realized in the Lorentzian case.

For the sake of definiteness and simplicity, we will concentrate mostly on the three-dimensional case. The discussion carries over virtually unchanged to $d = 4$ [5]. (Obviously, if these models yield sensible continuum theories, we expect them to be very different, one describing a topological quantum field theory, and the other a field theory of interacting gravitons.) The classical continuum action is simply Eq. (1), with $d = 3$. Each discrete Lorentzian space-time will be given by a sequence of two-dimensional compact spatial slices of fixed topology, which for simplicity we take to be that of a two-sphere. Each slice carries an integer time label t, so that the space-time topology is $I \times S^2$. The metric data will be encoded by triangulating this underlying space by three-dimensional simplices with definite edge length assignments. There are two types of edges: "spacelike" ones (of length squared $l^2 = a^2 > 0$, with the lattice spacing $a > 0$), which are entirely contained in a slice $t =$ const., and "timelike" ones (of length squared $l^2 = -\alpha a^2 < 0$), which start at some slice t and end at the next slice $t + 1$.

A metric space-time is built up by "filling in" for all times the three-dimensional sandwiches between t and $t + 1$. We consider only regular gluings which lead to simplicial *manifolds*. Our basic building blocks are given by three (m, n)-types of Lorentzian tetrahedra, where m and n denote the numbers of vertices the tetrahedron shares with the slices at t and $t + 1$, and $N_{31}(t)$, $N_{13}(t)$, and $N_{22}(t)$ their total numbers. Each triangulated space-time carries a discrete causal structure obtained by giving each timelike link an orientation in the positive t direction.

The discretized form of the Lorentzian action (1) is obtained from Regge's prescription for simplicial manifolds; see [5] for details. The contribution to the action from a single sandwich $[t, t + 1]$ is

$$\Delta S_\alpha(t) = 4\pi a k\sqrt{\alpha} + [N_{31}(t) + N_{13}(t)] \times (akK_1 - a^3\lambda L_1) + N_{22}(t)\,(akK_2 - a^3\lambda L_2)\,, \tag{2}$$

with the rescaled cosmological constant, $\lambda = k\Lambda$, and where

$$K_1(\alpha) = \pi\sqrt{\alpha} - 3\,\text{arcsinh}\,\frac{1}{\sqrt{3}\sqrt{4\alpha + 1}} - 3\sqrt{\alpha}\arccos\frac{2\alpha + 1}{4\alpha + 1}\,,$$

$$K_2(\alpha) = 2\pi\sqrt{\alpha} + 2\,\text{arcsinh}\,\frac{2\sqrt{2}\sqrt{2\alpha + 1}}{4\alpha + 1} - 4\sqrt{\alpha}\arccos\frac{-1}{4\alpha + 1}\,,$$

$$L_1(\alpha) = \frac{\sqrt{3\alpha + 1}}{12}\,, \qquad L_2(\alpha) = \frac{\sqrt{2\alpha + 1}}{6\sqrt{2}}\,.$$

Note that the sandwich action (2) already contains appropriate boundary contributions, such that S is additive under the gluing of contiguous slices.

At each time t the physical states $|g\rangle$ are parametrized by piecewise linear geometries, given by unlabeled triangulations g of S^2 in terms of equilateral Euclidean triangles. For a finite spatial volume N the number of states is exponentially bounded as a function of N and the orthogonal vectors $|g\rangle$ span a finite-dimensional Hilbert space $\mathcal{H}_N$. The transfer matrix $\hat{T}_N$ acts on the Hilbert space

$$H^{(N)} := \bigoplus_{i=N_{\min}}^{N} \mathcal{H}_i\,,$$

where $N_{\min}$ denotes the size of the minimal triangulation of the given topology ($N_{\min} = 4$ for S^2), and the states $|g\rangle$ will be normalized according to

$$\langle g_1 \,|\, g_2\rangle = \frac{1}{C_{g_1}}\,\delta_{g_1,g_2}, \qquad \sum_g C_g|g\rangle\langle g| = \hat{1}\,.$$

The symmetry factor C_g is the order of the automorphism group of the two-dimensional triangulation g, which for large triangulations is almost always equal to 1. With each step $\Delta t = 1$ we can now associate a transfer matrix $\hat{T}_N$ describing the evolution of the system from t to $t + 1$, with matrix elements

$$\langle g_2 | \hat{T}_N(\alpha) | g_1 \rangle \equiv G_\alpha(g_1, g_2; 1) = \sum_{\text{sandwich}(g_1 \to g_2)} e^{i\Delta S_\alpha}.$$

The sum is taken over all distinct interpolating three-dimensional triangulations of the "sandwich" with boundary geometries g_1 and g_2, each with a spatial volume $\leq N$. The propagator $G_N(g_1, g_2; t)$ for arbitrary time intervals t is obtained by iterating the transfer matrix t times, $G_N(g_1, g_2; t) = \langle g_2 | \hat{T}_N^t | g_1 \rangle$, and the infinite-volume limit is obtained by letting $N \to \infty$.

A brief remark is in order on our notion of time: The label t is to be thought of as the discretized analog of proper time. We do not claim that this is a physically distinguished notion of time, but it is nevertheless *a* possible choice, in the present case suggested by our regularization. In *continuum* formulations the proper time gauge is not usually considered, because it is a gauge choice that—considered for arbitrary geometries—goes bad in an arbitrarily short time. This problem does not occur in the discrete case: By construction we sum only over space-time geometries for which there is a globally well-defined (discrete) "proper time."

The action S associated with an entire space-time $S^1 \times S^2$ of length t in time direction is obtained by summing expression (2) over all $t' = 1, 2, \ldots t$ and identifying the two boundaries. The result is expressible as a function of three "bulk" variables, for example, the total numbers N_0 and N_3 of vertices and tetrahedra and t,

$$\begin{aligned} S_\alpha(N_0, N_3, t) = {} & N_0[4ak(K_1 - K_2) - 4a^3\lambda(L_1 - L_2)] + N_3(akK_2 - a^3\lambda L_2) \\ & + t\{4ak[\pi\sqrt{\alpha} - 2(K_1 - K_2)] + 8a^3\lambda(L_1 - L_2)\}. \end{aligned} \tag{3}$$

Because of the well-known inequality $N_0 \leq (N_3 + 10)/3$, valid for all closed three-dimensional simplicial manifolds, this implies the boundedness of the discretized Lorentzian action at fixed three-volume. We write the partition function as

$$Z_\alpha(k, \lambda, t) = \sum_{T \in \mathcal{T}_t(S^1 \times S^2)} e^{iS_\alpha[N_0(T), N_3(T), t(T)]}, \tag{4}$$

with $\mathcal{T}_t(S^1 \times S^2)$ denoting the set of all Lorentzian triangulations of $S^1 \times S^2$ of length t. A necessary condition for the existence of a meaningful continuum limit is the exponential boundedness of the number of possible triangulations as a function of N_3. In our case, this follows trivially from the same property for Euclidean triangulations [6,7], since the Lorentzian space-times form a subset of the former. Note that the convergence of the partition function implies the absence of divergent "conformal modes."

As it stands, the sum (4) over complex amplitudes has little chance of converging, due to the contributions of an infinite number of triangulations with arbitrarily large volume N_3. In order to make it well defined, one must perform a Wick rotation, just as in ordinary quantum field theory. Thanks to the presence of a distinguished global time variable in our model, we can associated a unique Euclidean triangulated space-time with every Lorentzian history contributing in (3). It is obtained by taking the *same* topological triangulation and changing the squared lengths of all timelike edges from $-\alpha a^2$ (Lorentzian) to αa^2 (Euclidean), leaving the spacelike edges unchanged. We can then use Regge's prescription for calculating the (real) Euclidean action $S_\alpha^E(N_0, N_3, t)$ associated with the resulting Euclidean metric space-time (where α is always taken to be positive). After some algebra one verifies that by a suitable analytic continuation in the complex α plane from positive to negative real α, the Euclidean and Lorentzian actions are related by

$$S_{-\alpha}(N_0, N_3, t) = iS_\alpha^E(N_0, N_3, t), \tag{5}$$

for $\alpha > \frac{1}{2}$. For $\alpha = 1$ in (5) one rederives the familiar expression employed in equilateral Euclidean dynamical triangulations, namely,

$$\begin{aligned} \frac{1}{i} S_{-1} \equiv S_1^E = {} & -ak(2\pi N_1 - 6N_3 \arccos\tfrac{1}{3}) \\ & + a^3\lambda N_3 \frac{1}{6\sqrt{2}}. \end{aligned} \tag{6}$$

Our strategy for evaluating the partition function is now clear: for any choice of $\alpha > \frac{1}{2}$, continue (3) to $-\alpha$, so that

$$\sum_{T \in \mathcal{T}_t(S^1 \times S^2)} e^{iS_\alpha(N_0, N_3, t)} \xrightarrow{\alpha \to -\alpha} \sum_{T \in \mathcal{T}_t(S^1 \times S^2)} e^{-S_\alpha^E(N_0, N_3, t)}. \tag{7}$$

Because of the exponential boundedness, the Wick-rotated Euclidean state sum in (7) *is* now convergent for suitable choices of the bare couplings k and λ. We can therefore proceed in two ways: either attempt to perform the sum analytically, by solving the combinatorics of possible causal gluings of the tetrahedral building blocks (as has been done in $d = 2$ [2]) or use Monte Carlo methods to simulate the system at finite volume. Once the continuum limit has been performed, we can rotate back to Lorentzian signature by an analytic continuation of the continuum proper time T to iT. If we are interested only in vacuum expectation values of time-independent observables and the properties of the Hamiltonian, we do not need to perform the Wick rotation explicitly, just as in usual Euclidean quantum field theory.

Let us now establish some properties of the discrete real transfer matrix $\hat{T} \equiv \hat{T}(\alpha = -1)$ of our model that are necessary for the existence of a well-defined Hamiltonian $\hat{h}$, such that $\hat{T} = e^{-a\hat{h}}$. These will be useful in any proof of the existence of a self-adjoint *continuum* Hamiltonian $\hat{H}$. It is difficult to imagine boundedness and positivity arising in the limit from regularized models without these properties. In our case, $\hat{T}_N$ is symmetric, bounded for finite spatial volume N, and strictly positive. The only nontrivial property to show is strict positivity. Positivity, $\hat{T}_N \geq 0$, follows from the reflection positivity of our model under reflection with respect to planes of constant t, for both integer and half-integer t [5] (see also [8]). However, for the existence of a Hamiltonian we must show that zero cannot occur as an eigenvalue. It suffices to show that

$$\langle g_1|\hat{T}_N^2|g_1\rangle \equiv \sum_g C_g \langle g_1|\hat{T}_N|g\rangle\langle g|\hat{T}_N|g_1\rangle > 0 \tag{8}$$

for all states $|g_1\rangle$. From this it would follow that $\hat{T}_N^2 > 0$, which together with the positivity $\hat{T}_N \geq 0$ would imply the desired result. Since the right-hand side of (8) is a sum of positive terms, we must show that for each $|g_1\rangle$ there is at least one state $|g\rangle$ with $\langle g|\hat{T}_N|g_1\rangle > 0$. It is straightforward to show that a possible choice is given by $|g\rangle = |g_{\min}\rangle$, where $g_{\min}$ is the minimal triangulation of two-volume 4 of the two-sphere. We deduce that the transfer matrix is strictly positive, $\hat{T}_N > 0$, and that for finite triangulations there is a self-adjoint Hamiltonian operator $\hat{h}_N$ which is bounded from below.

It should be emphasized that although the summation in the path integral is performed in the "Euclidean sector" of the theory, our construction is not *a priori* related to any path integral for Euclidean gravity proper. The point, already made in the two-dimensional case [2], is that we sum only over a selected class of geometries, which are equipped with a causal structure. Such a restriction incorporates the Lorentzian nature of gravity and has no analog in Euclidean gravity. We therefore expect our Lorentzian statistical mechanics model to have a totally different phase structure from that of Euclidean dynamical triangulations. This expectation is corroborated by an analysis of the "extreme phases" of Lorentzian quantum gravity, to determine which configurations dominate the path integral

$$Z_\alpha^E(k, \lambda, t) = \sum_{T \in \mathcal{T}_t(S^1 \times S^2)} e^{-S_\alpha^E}, \tag{9}$$

for either very small or very large $k > 0$. To make a direct comparison with the Euclidean analysis [9,10], we set without loss of generality $\alpha = 1$ in Eq. (9) and rewrite the Euclidean action (6) as $S_1^E = k_3 N_3 - k_1 N_1$. In the thermodynamic limit $N_3 \to \infty$, and assuming a scaling behavior such that $t/N_3 \to 0$, one derives kinematical bounds on the ratio of links and tetrahedra, $\xi := N_1/N_3$, namely,

$$1 \leq \xi \leq \tfrac{5}{4}.$$

This is to be contrasted with the analogous result in the Euclidean case, where $1 \leq \xi \leq \frac{4}{3}$. It implies that the branched-polymer (or stacked-sphere) configurations, which are precisely characterized by $\xi = \frac{4}{3}$, and which dominate the Euclidean state sum at large k_1, cannot be realized in the Lorentzian setting. The opposite extreme, at small k_1, is associated with the saturation of the inequality

$$N_1 \leq N_0(N_0 - 1)/2, \tag{10}$$

and in the Euclidean theory goes by the name of "crumpled phase." At equality, every vertex is connected to every other vertex, corresponding to a manifold with a very large Hausdorff dimension. Again, it is impossible to come anywhere near this phase in the continuum limit of the Lorentzian model. Instead of (10), we have now separate relations for the numbers $N_1^{(\mathrm{sl})}$ and $N_1^{(\mathrm{tl})}$ of space- and timelike edges,

$$N_1^{(\mathrm{sl})} = \sum_t [3N_0(t) - 6] = 3N_0 - 6t,$$
$$N_1^{(\mathrm{tl})} \leq \sum_t N_0(t) N_0(t+1). \tag{11}$$

Assuming canonical scaling, the right-hand side of inequality (10) behaves like $(\text{length})^6$, whereas the second relation in (11) scales only like $(\text{length})^5$.

We conclude that the phase structure of Lorentzian gravity must be very different from that of the Euclidean theory and that the extreme branched-polymer and crumpled configurations cannot occur. This is another example of causal structure acting as a "regulator" of geometry. It also raises the hope that the mechanism governing the phase transition will be different and potentially lead to a nontrivial continuum theory, in three as well as in four dimensions.

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[1] J. Ambjørn, J. Jurkiewicz, and R. Loll, hep-th/0001124; R. Loll, Living Reviews in Relativity 1998-13, http://www.livingreviews.org/Articles/Volume1/1998-13loll
[2] J. Ambjørn and R. Loll, Nucl. Phys. **B536**, 407 (1998).
[3] J. Ambjørn, K. N. Anagnostopoulos, and R. Loll, Phys. Rev. D **60**, 104035 (1999).
[4] J. Ambjørn, K. N. Anagnostopoulos, and R. Loll, Phys. Rev. D **61**, 044010 (2000).
[5] J. Ambjørn, J. Jurkiewicz, and R. Loll (to be published).
[6] M. Carfora and A. Marzuoli, J. Geom. Phys. **16**, 99 (1995); J. Math. Phys. (N.Y.) **36**, 6353 (1995).
[7] J. Ambjørn, M. Carfora, and A. Marzuoli, *The Geometry of Dynamical Triangulations,* Lecture Notes in Physics Vol. 50 (Springer, Berlin, 1997).
[8] I. Montvay and G. Münster, *Quantum Fields on a Lattice* (Cambridge University Press, Cambridge, England, 1994).
[9] D. Gabrielli, Phys. Lett. B **421**, 79 (1998).
[10] J. Ambjørn *et al.,* Nucl. Phys. **B542**, 349 (1999).