Nonlinear Cyclotron Resonant Wave-Particle Interaction in a Nonuniform Magnetic Field

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Quasilinear analysis of wave-particle interactions is presented for plasma flowing in a weakly nonuniform magnetic field configuration. The method presented is based on a scale separation between the length scales of quasilinear relaxation and the magnetic field inhomogeneity, allowing one to obtain large scale solutions for both particle distribution functions and wave spectra, without going into the details of the small scale quasilinear relaxation. The numerical example shows the existence of a secondary instability for an initially stable particle distribution function.

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The quasilinear theory of wave-particle interactions was originally formulated in a homogeneous plasma approximation [1–4]. It was discovered soon afterwards that even a small inhomogeneity of the plasma parameters leads to significant changes in the nature of the quasilinear relaxation process, due to effects such as breaking of the resonance conditions (resonance broadening) and limits on the time of resonant interaction [5].

There is large class of problems, arising mostly in astrophysical contexts, where the plasma with inhomogeneous parameters is embedded in a weakly nonuniform magnetic field. Physical examples include configurations used to model solar coronal heating and solar wind acceleration [6]. In particular, the recent observations by the Solar and Heliospheric Observatory of preferential ion heating and acceleration [7] provide an extensive framework for development of coronal heating models and application of the method developed below.

These problems can be described as having two length scales of evolution: (i) the scale of the magnetic field inhomogeneity and (ii) the scale of quasilinear relaxation, that is, of the order of several Larmor radii. The ratio of the scales in some cases may be quite large (10^5–10^6). For more than 20 years these problems were treated either by application of homogeneous quasilinear diffusion analysis or by direct particle simulations. In both cases the solutions obtained were strictly applicable only to small regions, with scales comparable to or only slightly exceeding the typical scales of homogeneous quasilinear relaxation. No large scale solutions have as yet been presented.

The purpose of this Letter is to outline in simplified form a method for deriving and obtaining a solution for the large scale equations. (The preliminary application of the method to solar wind acceleration and heating was reported at the Fall'98 American Geophysical Union (AGU) meeting [8] and adopted at the Fall'99 AGU [9].)

We consider a system consisting of protons and some resonant particles (either tail protons or ions) with density n_0 and mass $\tilde{m} m_p$ (m_p is the proton mass). All particles enter a region with a nonuniform magnetic field and have some average speed U_0 directed along the magnetic field (U_0 considerably exceeds the thermal velocity v_T of the particles). For the sake of simplicity we assume that all plasma parameters depend only on one coordinate, z, directed along the external magnetic field $B_z(z)$. We introduce a parameter characterizing the ratio of the relevant scales, $\lambda = \Omega_0 z_0 / U_0 \gg 1$, where $\Omega_0 = eB_0/m_p c$ is the proton cyclotron frequency at the point of entry [$B_0 = B_z(0)$], and z_0 is some characteristic scale of magnetic field inhomogeneity $z_0 \sim |(1/B_z)(dB_z/dz)|^{-1}$. We also assume that some particles are in cyclotron resonance with Alfvén waves injected into the plasma flow and that the waves have an initially wide spectrum with a power law distribution. The waves are moving with phase speed $v_w(z)$ with respect to the plasma. The spectrum comprises wavelengths that are small in comparison to the inhomogeneity scale z_0, so that the approximation of geometrical optics can be applied.

We will choose the energy $u = 2E = (v_\perp^2 + v_z^2)$ and the perpendicular magnetic moment $\mu = v_\perp^2/2B_z$ as independent variables instead of $v_\parallel$, $v_\perp$, and will use dimensionless variables, z, $v_w(z)$, u, and μ, normalized by z_0, U_0, U_0^2, and U_0^2/B_0, respectively. We also introduce dimensionless magnetic field variables $B^\pm = (B_x \pm iB_y)/B_0$, the magnetic field Fourier mode $b^\pm(\omega, k)$, and phase $\phi^\pm(\omega, k, z, u, \mu)$ of particle rotation in a wave field with frequency ω and wave number k. The phase can be calculated, using a system of reference moving with particle speed v_z, as a difference in particle rotation and magnetic field vector rotation,

$$\phi^\pm(\omega, k) = -\int_{z_0}^{z} \frac{dz}{v_z} [\omega - k v_z \pm \Omega_h]. \qquad (1)$$

The linear response of the particle distribution function f_1 to the presence of waves can be obtained by integrating the particle kinetic equation along the zero order trajectories and asymptotically expanding the resulting integral afterwards. The result may be written as (we omit the derivations here and below to conserve space; see [10,11] for derivations of similar expressions)

$$f_1^\pm(\omega, k) = i\frac{\sqrt{\lambda}}{2}\left\{\frac{1}{\sqrt{|\phi_{zz}^\pm|}}\frac{\mu}{\sqrt{2\mu B_z}}\hat{L} f_0\right\} \times [b^\pm(\omega, k) e^{i(\lambda\phi^\pm + \Delta^\pm \pi/4)} - \text{c.c.}], \quad (2)$$

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where $\hat{L}$ is the operator

$$\hat{L} = \left(1 - \frac{v_w}{\sqrt{u - 2\mu B_z}}\right)\frac{\partial}{\partial \mu} - \frac{2 v_w B_z}{\sqrt{u - 2\mu B_z}}\frac{\partial}{\partial \mu}, \tag{3}$$

and $\phi_z^{\pm} = \partial\phi^{\pm}/\partial z$, $\phi_{zz}^{\pm} = \partial^2\phi^{\pm}/\partial z^2$, $\Delta^{\pm} = \mathrm{sgn}(\phi_{zz}^{\pm})$.

The equation of quasilinear diffusion can be obtained (see [10,11]) as

$$\frac{\partial f_0}{\partial z} + \frac{\lambda^{3/2}}{2\sqrt{2\pi}} \int dk\, d\omega\, \delta(\omega - \omega_k) \times \hat{L}\left\{\frac{|b(\omega,k)|^2}{\sqrt{\phi_{zz}^{\pm}}}\frac{\mu}{B_z}\hat{L} f_0\right\} = 0. \tag{4}$$

The delta function $\delta(\omega - \omega_k)$ restricts the integration over ω to propagating waves, that is, to waves that satisfy the local dispersion relation $\omega_k = \omega(k,z)$.

The equation for the particle distribution function f should be supplemented by the corresponding equation for the evolution of the wave spectrum $|b(\omega,k)|^2$. Using the geometrical optics approximation it can be written (see [11]) as

$$\frac{\partial\omega}{\partial k}\frac{\partial|b|^2}{\partial z} - \frac{\partial\omega}{\partial z}\frac{\partial|b|^2}{\partial k} = \lambda^{3/2}\gamma^{\pm}(\omega,k)\,|b|^2, \tag{5}$$

where $\gamma^{\pm}(\omega,k)$ is growth or damping rate due to resonant interaction with the particles. It can be obtained from the linear theory using (2) (see [11]) as

$$\gamma^{\pm} = \frac{\eta_0 v_w B_z}{2\sqrt{2\pi}} \int \frac{\mu}{\sqrt{u - 2\mu B_z}} \frac{\hat{L} f_0}{\sqrt{|\phi_{zz}^{\pm}|}}\, d\mu\, du, \tag{6}$$

where η_0 is the ratio of the particle energy to the wave energy at $z = z_0$.

Equations (4) and (5) describe the self-consistent evolution of particles and waves due to resonant interaction in a plasma in a nonuniform magnetic field. The large parameter $\lambda^{3/2}$ in these equations simply confirms that quasilinear evolution is a much faster process than the changes induced by weak inhomogeneity of the medium. It is the presence of this large coefficient that makes a direct solution of this system of equations practically impossible.

To develop a practical method of treating this system of equations we must find a way to avoid resolving the microscopic dynamics of quasilinear relaxation on short length scales. First, note that the total energy flux is conserved. Hence, integrating (4) and (5) we may write the energy flux conservation as

$$\int\left[\frac{\partial\omega}{\partial k}\frac{\partial|b|^2}{\partial z} - \frac{\partial\omega}{\partial z}\frac{\partial|b|^2}{\partial k}\right] dk\, d\omega\, \delta(\omega - \omega_k) = -\frac{\eta_0}{2}\frac{\partial}{\partial z} B_z \int d\mu\, du\, u f_0. \tag{7}$$

Although this expression represents conservation of the energy flux integrated over the entire wave spectrum, we may assume that it is valid for every spectral subinterval or for the spectral density of the energy flux. Indeed, at every distance z the interaction between the particles and the waves takes place only in the vicinity of the local resonance. Hence, we may use the condition of the local resonance $\phi_z^{\pm} = 0$ to transform the integral over u in (7) to an integral over ω. We obtain for $|b(\omega,z)|^2$

$$|b|^2 = \frac{\eta_0 B_z}{2} \int d\mu \left(\frac{\partial\phi_z^{\pm}}{\partial\omega} \Big/ \frac{\partial\phi_z^{\pm}}{\partial u}\right) u f_0 + C, \tag{8}$$

where u should be obtained from the resonant condition $\phi_z^{\pm} = 0$.

Although Eq. (8) does not have the small scales anymore, it cannot be solved directly, because it includes an unknown distribution function f_0. Taking into account that the magnetic field is only weakly nonuniform/inhomogeneous, we can substitute the stationary solution of the uniform/homogeneous quasilinear equation in place of f_0, that is, the solution of the equation

$$|b(\omega,k)|^2\hat{L}\tilde{f}_0 = 0. \tag{9}$$

Using the variables $w = u - 2v_w\sqrt{u - 2\mu B_z}$ and $v_z = \sqrt{u - 2\mu B_z}$, it is easy to show that the solution of the above stationary problem can be written as a function which is constant along the lines of pitch angle diffusion $w = v_\perp^2 + (v_z - v_w)^2$ (see [3]). In this case, we can write $\tilde{f}_0 = \tilde{f}_0(z,w)$ and, after changing the variable of integration in (8), we get

$$|b|^2 = C + \frac{\eta_0 v_w}{4\omega^2 B_z}\left[\pm\int_{w_0}^{\infty} dw\, w\tilde{f}_0 \pm v_w^2\left(1 \pm \frac{2B_z}{\tilde{m}\omega}\right)\int_{w_0}^{\infty} dw\,\tilde{f}_0\right], \tag{10}$$

where $w_0 = (B_z v_w/\tilde{m}\omega)^2$. Therefore, if the wave energy is sufficiently large the change of wave energy in the high frequency part of the spectrum ($\omega \to \infty$) will be proportional to ω^{-2}.

The stationary solution will form in the entire area of resonance only if there is enough energy contained in every part of the wave spectrum. If this is not the case, the quasilinear diffusion will stop before forming a "plateau" along the lines of diffusion w in the whole region. Equation (9) will still be satisfied, because $|b(\omega,k)|^2$ becomes zero in parts of the spectrum with insufficient energy. Independent plateaus will be formed between each of these insufficient energy regions. In this case we can create the procedure for obtaining the solution numerically.

First of all, we introduce new coordinates w,ψ directed perpendicular and parallel to the lines of diffusion,

$$v_\perp = \sqrt{w}\sin\psi, \qquad v_z = \sqrt{w}\cos\psi + v_w. \tag{11}$$

In these coordinates the solution for the plateau distribution function $\tilde{f}_0$ can be found from the conservation of the particle density flux for every value of w and in every interval (ψ_1,ψ_2),

$$\tilde{f}_0(z,w) = \frac{1}{A(\psi_2) - A(\psi_1)} \times \int_{\psi_1}^{\psi_2} d\psi\, v_\perp(w,\psi) v_z(w,\psi) f_0, \tag{12}$$

where the normalization factor $A(\psi) = -1/2w\cos^2\psi - \sqrt{w}\,v_w\cos\psi$. The limits of integration $\psi_{1,2}$ are functions of z and v_z (or ω) and should be determined either from the boundaries of the resonance region or from the frequencies where $|b(\omega,k)|^2$ becomes equal to zero.

Now we will use (12) to obtain the procedure for determining f_0 simply by advancing in z starting from $z = 1$ and using some step Δz. Writing Taylor expansion of $f_0(z,w,\psi)$ and substituting it into (12) we will get

$$\tilde{f}_0(z,w) = \frac{1}{A(\psi_2) - A(\psi_1)} \times \int_{\psi_1}^{\psi_2} d\psi\, v_\perp v_z f_0(z-\Delta z, w, \psi) + \Delta z \int_{\psi_1}^{\psi_2} d\psi\, v_\perp v_z \frac{\partial f_0}{\partial z} + \ldots . \quad (13)$$

Substituting the derivative $\partial f_0/\partial z$ from the quasilinear equation (4) and taking into account that the operator $\hat{L}$ in w, ψ coordinates has the very simple form,

$$\hat{L} = -\frac{B_z}{v_\perp v_z}\frac{\partial}{\partial \psi}, \quad (14)$$

we can easily find that the last term in (13) is equal to zero (the limits of integration correspond to $|b(\omega,k)|^2 = 0$ or $\mu = 0$).

We once again use the weak nonuniformity of the medium and replace the exact distribution function $f_0(z-\Delta z, w, \psi)$ in (13) by the homogeneous solution formed on the previous step $\tilde{f}_0(z-\Delta z, w, \psi) = \tilde{f}_0(z - \Delta z, w(z-\Delta z))$. After that, we can write the system of equations for the large scale evolution as

$$|b|^2 = \frac{\eta_0 B_z}{2}\int d\mu \left(\frac{\partial \phi_z^\pm}{\partial \omega} \Big/ \frac{\partial \phi_z^\pm}{\partial u}\right) u \tilde{f}_0(z,\mu,u) + C, \quad (15)$$

$$\tilde{f}_0(z) = \frac{1}{A(\psi_2) - A(\psi_1)}\int_{\psi_1}^{\psi_2} d\psi\, v_\perp v_z \tilde{f}_0(z-\Delta z). \quad (16)$$

Note that we normalized z by the scale of the inhomogeneity z_0. Therefore the condition of small step size $\Delta z \ll 1$ means that it should be small in comparison with the inhomogeneity scale and may be comparable or even larger than the scale of quasilinear relaxation.

As an example, we obtained a numerical solution of system (15)–(16) for a weakly nonuniform magnetic field of the form $B_z(z) = B_z(1)/z^2$ $(z \geq 1)$. The phase speed of the waves was also taken to be a function of distance, $v_w(z) = U_0 + v_w^0/z$ (where $v_w^0/U_0 = 1.5$). This demonstration model can be considered as an oversimplified model of resonant interaction of heavy ions or tail protons with Alfvénic turbulence in the solar wind stream. Because of the large difference of scales of inhomogeneity and quasilinear diffusion the solution looks like a wave slowly moving on the particle distribution function with instant pitch-angle diffusion taking place at the front of the wave. Figure 1 shows the evolution of the particle distribution function $f(z)$ with distance at four different positions. The discontinuity in the distribution function is not a numerical artifact. It usually exists in solutions of homogeneous quasilinear equations as well and can be explained by the finite size of an area of the resonance. The corresponding plots of the wave spectral density $|b(\omega,z)|^2$ are shown in the next figure (Fig. 2). The initial wave spectrum is taken to be proportional to ω^{-1}. The resonant interaction results in the evolution of both the distribution function and the high frequency part of the wave power spectrum. The example exhibits one very interesting feature. As one can see from Fig. 2, in addition to damping of high frequency waves, the inhomogeneous quasilinear relaxation may result in wave growth, although a locally homogeneous quasilinear analysis of the distribution function in all panels of Fig. 1 leads to the conclusion that the distribution function is stable. In order to understand the nature of this secondary instability, we should take into account that the wave phase speed decreases with distance. Therefore, the particles located in the left part of

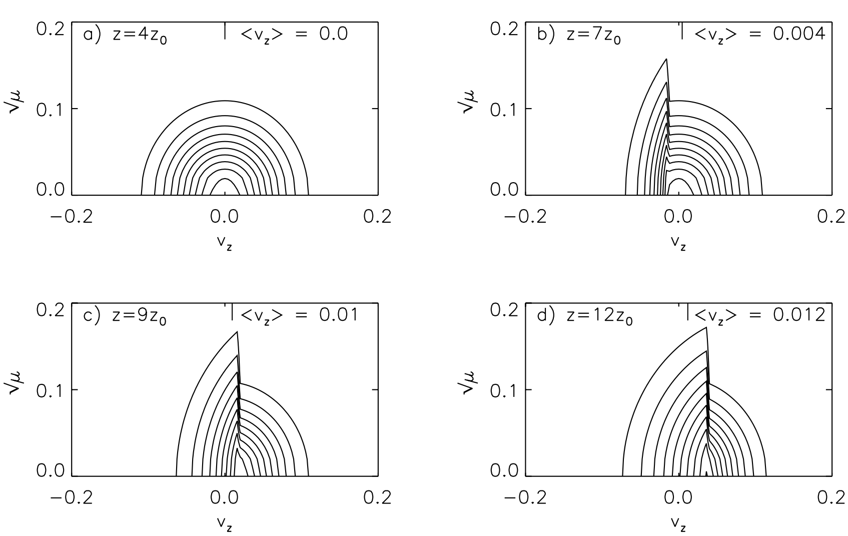

FIG. 1. Contour lines of particle distribution function $f(x)$ in the $(v_z, \sqrt{\mu})$ plane at four different distances. The left part of the distribution function in (b)–(d) represents a quasilinear plateau formed due to wave-particle interaction ($\langle v_z \rangle$ is the average velocity of the distribution).

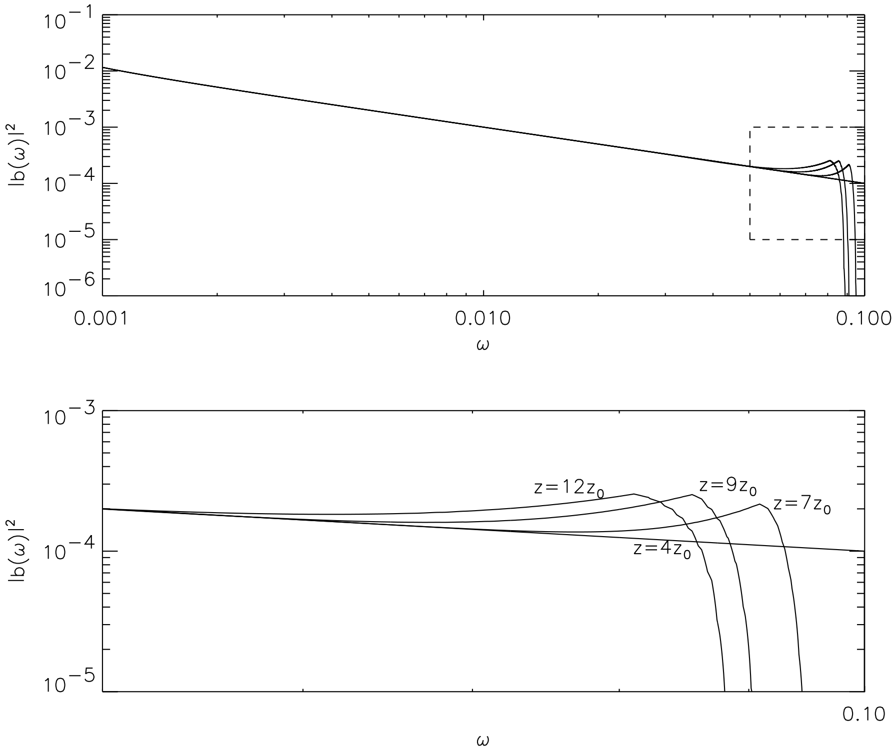

FIG. 2. Wave spectral density $|b(\omega)|^2$ plotted at the same distances as in the previous figure. The lower panel is a detailed plot of the boxed part of the upper panel. The frequency ω normalized to the initial cyclotron frequency Ω_0. The initial wave spectrum is $\sim\omega^{-1}$.

the distribution function in panels (b)–(d) (Fig. 1), where the quasilinear plateau has already been formed, will fall in resonance with waves having smaller phase velocity as they move further downstream, and can transfer their energy to the waves.

In conclusion we have presented an analysis based on a scale separation between the length scales of quasilinear relaxation and the magnetic field inhomogeneity, allowing one to obtain large scale solutions for both particle distribution function and wave spectral density, without going into the details of the small scale quasilinear relaxation. We should note that the method is similar to a bounce averaged quasilinear diffusion approach [10], although the absence of a large scale quasiperiodicity (absence of turning points) in a divergent magnetic field dictates a completely different treatment in this case of large scale inhomogeneity. We have also outlined a method for numeric solution of the large scale equations of quasilinear relaxation in a weakly nonuniform medium. The method can be expanded to be applicable to any problem of cyclotron wave-particle interaction in inhomogeneous plasma, including the solar wind acceleration and heating. The numerical model shows a new effect; the distribution function which is locally stable shows secondary instability due to the collective input of resonances and inhomogeneous streaming. This instability will be investigated separately.

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