Automatic Quantum Error Correction

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Criteria are given by which dissipative evolution in the weak coupling limit can transfer populations and coherences between quantum subspaces, without a loss of coherence. The result is a form of quantum error correction implemented by the joint evolution of a system with a cold bath. It requires no external intervention and, in principle, no ancilla. An example of such a system, consisting of three dipole ordered spin 1/2 particles in a resonator, is given. The qubit, or the triple quantum coherence of the spins, is protected against all spin-flip errors.

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Quantum computation is of interest because algorithms have been discovered with a significant speedup over any classical algorithm [1,2], although these may be unique cases [3]. A quantum computation can be protected against decoherence either passively [4] or actively, by quantum error correcting codes (QECC). Codes have been devised [1,5–7] and experimentally demonstrated [8] that can protect a set of states, the code words, against a given set of entanglements with the environment [9]. QECC is similar to a quantum erasure experiment [10], except that one cannot manipulate the environment. Instead, one disentangles the code words from the environment by transferring the entanglement to other degrees of freedom.

Current proposals [5,7] implement QECC in a two-step process. First, the information about which error has struck must be stored in a set of qubits (two level systems) called the ancilla. The ancilla must be accessible to manipulation or measurement by the programmer, and they must initially be in a state of zero entropy [11]. In the last step, this information is used to repair the error. A difficulty with this scheme is that there is a high premium placed on using as few qubits as possible, because the number of transitions to be manipulated, unwanted thermal effects [12], and decoherence rates [13] all increase exponentially with system size. But, to take an example, fault-tolerant QECC of even a single qubit can require 15 physical qubits, ten of which must be in a known state [6]. And as many as 28 coherent manipulations of qubit pairs are required for each repair cycle, because the eigenvalues of operators such as $I_{x1}I_{x2}I_{z3}I_{z5}$ must be placed in the ancilla (see Fig. 2 of Ref. [6]). Although the efficiency of QECC improves for larger computations, a physical scale-up factor of 22 is still required to factorize a thousand-digit number [14].

We discuss here another approach, called "automatic quantum error correction" (AQEC) because no intervention by the programmer is required to implement it. Instead, error correction occurs via the joint evolution of a system interacting with a cold bath in the weak coupling limit. Although the two-step implementation can also be formulated as a dissipative dynamics [15], we present a distinct method that makes no use of "which error" information. *Thus, no coherent manipulations and, in principle, no ancilla are necessary to implement AQEC.* (Here, the bath refers to degrees of freedom that must be cooled, but need not be otherwise manipulated, in contrast to the ancilla, while the programmer has no control over the environment.)

Clearly, a dissipative quantum system can store classical bits of information. It is less obvious that such systems can store qubits, since dissipation usually destroys coherence. The key ideas are to (1) find a system where errors always add energy to the system, and to (2) remove the excitation from distinct code words and their excited states in a symmetric manner. This prevents the bath from gaining information on which code words are occupied by the system, and thus avoids entanglements with the bath. Below, we give conditions necessary to implement this, and show how they give rise to a dissipative evolution that becomes a recovery superoperator. The last section proposes a physically realizable system that protects a qubit against spin-flip errors.

Conditions for AQEC.—Consider a finite dimensional system with density matrix ρ and Hamiltonian H_s, interacting with a bath in thermal equilibrium ρ_b with Hamiltonian H_b through the operator V. The first assumption is that the bath and system are in the weak coupling limit:

$$\frac{d}{dt}\rho = L\rho = -i[H_s,\rho] + K^{\natural}\rho\,, \tag{1}$$

$$K = -\mathrm{tr}_b\left\{\int_0^\infty [V,[V(t),\rho\otimes\rho_b]]\,dt\right\}, \tag{2}$$

$$V(t) = e^{-i(H_s+H_b)t}Ve^{i(H_s+H_b)t}, \tag{3}$$

$$K^{\natural} = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{iH_st}Ke^{-iH_st}\,dt\,, \tag{4}$$

as derived by Davies [16]. L is a superoperator [17] that has the standard form of a generator of a quantum dynamical semigroup, $R(t) = \exp(Lt)$ [18].

The assumption of the weak coupling limit is vital to the success of AQEC. Careful discussions of these equations

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are found in Refs. [13,18]. They have often been successfully employed [18,19] to describe relaxation in quantum optics [20] and magnetic resonance [21], suggesting that such baths are not uncommon in nature. (But some previous derivations have been incorrect; see Ref. [22].) Even if small memory effects are present, L is still expected to be a good approximation to the system dynamics because the covergence of the dynamical generator to L is uniform [16].

Given a set of errors, the goal is to find conditions on H_s and V so that $R(t)$ approaches a recovery superoperator for large t [9]. In order to state these conditions, define $|\phi_n^i\rangle$ to be an orthonormal set, $\langle\phi_n^i \mid \phi_m^j\rangle = \delta_{ij}\delta_{nm}$, of $N(M+1)$ states, where $1 \leq i \leq N$ and $0 \leq n \leq M$. There are N code words, the states $|\phi_0^i\rangle$. Associated with each code word is an M dimensional subspace, $|\phi_n^i\rangle$ with $n > 0$, called the funnel for i. The following projection operators will be useful:

$$P_n = \sum_{i=1}^{N} |\phi_n^i\rangle\langle\phi_n^i|, \tag{5}$$

$$F_i = \sum_{n=1}^{M} |\phi_n^i\rangle\langle\phi_n^i|, \tag{6}$$

$$Q = \sum_{n=0}^{M} P_n = P_0 + \sum_{i=1}^{N} F_i\,. \tag{7}$$

P_0 is the projection operator into the coding space, and F_i projects into the funnel subspace associated with code word i. Q is the projector into a proper subspace of the entire system, within which all the dynamics is expected to occur. The funnels are mutually disjoint, $F_iF_j = \delta_{ij}F_i$, and separate from the coding space, $P_0F_i = F_iP_0 = 0$. The F_i will play the role of a funnel for each code word: any error will be required to transfer amplitude from $|\phi_0^i\rangle$ only into F_i, where the dissipative dynamics will draw it back down to the original code word.

Define Ω to be the lowest eigenstate of H_s outside of Q that has a nonzero matrix element of V with a state in Q. Demand that H_s can be put into the following form:

$$H_s = \sum_{n=0}^{M} \omega_n P_n + (1-Q)H_s(1-Q), \tag{8}$$

$$\omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{M-1} < \omega_M < \Omega\,, \tag{9}$$

so there is a degeneracy between the code words and the funnels, which makes the dynamics of separate funnels appear indistinguishable to the bath. This allows dissipation to transfer coherences. The effect is similar to phase matching in nonlinear optics [23]. For a simple case, it can be shown that a mismatch of $\Delta\omega$ between two funnels degrades the coherence transfer by $\Delta\omega^2$ [24]. Note that the degeneracy need not be a global symmetry of the system. Also, assume the $\omega_n - \omega_m$ are distinct, so that they excite orthogonal bath modes.

Suppose the initial environmental state is $|e_0\rangle$. The argument can be extended to impure states by summation over all environmental states. An error $E = \sum_a A_a(|e_a\rangle\langle e_0| + |e_0\rangle\langle e_a|)$ is a unitary transform, with system operators A_a. When $|\langle e_b \mid e_a\rangle| < 1$, the environment becomes entangled with the system. AQEC requires the following:

$$(1 - F_i)A_a|\phi_0^i\rangle = 0\,, \tag{10}$$

$$\forall\ n > 0, \qquad \langle\phi_n^i|A_a|\phi_0^j\rangle = \delta_{ij}\alpha_{an}\,, \tag{11}$$

where the constants α_{an} must be independent of i. Thus, errors drive code word populations only into their respective funnels. The bath should be unable to distinguish separate funnels and code words, so

$$\langle\phi_0^i|V|\phi_0^j\rangle = 0\,, \tag{12}$$

$$\forall\ n > m \geq 0, \qquad \langle\phi_m^i|V|\phi_n^j\rangle = \delta_{ij}\nu_{nm}\varphi_{nm}^\dagger\,, \tag{13}$$

where the bath mode annihilation operators φ_{nm} and the constants ν_{nm} are independent of i. Thus, V does not switch amplitude between separate funnel and code word subspaces, and acts symmetrically on the separate funnels. This is not a very strict requirement, since code words have the property that they be macroscopically indistinguishable to environmental interactions [25]. In order to remove all the excited population, there should always be some transition driven by V downward from any funnel state:

$$\forall\ n > 0, \qquad \exists\ m \quad \nu_{nm} \neq 0\,. \tag{14}$$

Finally, the bath should be able to accept excitation at all the system transitions, and it should be this cold:

$$kT_b/\hbar \ll |\omega_n - \omega_m|, \qquad |\Omega - \omega_n|\,. \tag{15}$$

Altogether, there is the assumption of a weak coupling limit in Eq. (1), and the criteria of Eqs. (8)–(15).

Can P_0 hold information, even in the absence of errors, given that the bath is always present in AQEC? In order to do so, P_0 must form a decoherence free subspace with respect to V [4]. Equation (12) satisfies some of the conditions [4], but Eq. (13) does not for all possible baths. However, Eq. (15) and the weak coupling limit imply that the bath does not possess sufficient energy to excite a state from P_0. So for a cold enough bath, P_0 is decoherence-free with respect to V.

Next, we must find whether it is possible to recover the corrupt information. We must see if the above satisfies the criteria of QECC [9]. From Eq. (6) and (10), it follows that $F_iA_a|\phi_0^j\rangle = \delta_{ij}A_a|\phi_0^j\rangle$. Then from Eq. (11),

$$\langle\phi_0^i|A_a^\dagger A_b|\phi_0^j\rangle = \delta_{ij}\sum_{n,m}\alpha_{an}^\star\alpha_{bm}\,. \tag{16}$$

Since the α_{an} are independent of i, the result follows. The reverse implication does not hold. QECC allows errors to transform the code word amplitudes in distinct ways, since it can make use of the information about which error occurred in order to repair the error.

Finally, we must find $R(t)$. Before an error, the system is in P_0, and after, from Eq. (10), it is in the F_i. Because of Eqs. (9) and (15), only downward transitions in energy driven by V are allowed, so all the system dynamics will occur in Q. Use Eqs. (12) and (13) to write

$$QVQ = \sum_{n>m\geq 0}^{M} (\nu_{nm} G_{nm} \varphi_{nm}^{\dagger} + \nu_{nm}^{\star} G_{nm}^{\dagger} \varphi_{nm}),$$
$$\forall\ n > m \geq 0, \qquad G_{nm} = \sum_{i=1}^{N} |\phi_m^i\rangle\langle\phi_n^i|. \tag{17}$$

G_{nm} is a lowering operator between degenerate subspaces of the system. Because of Eq. (15), the bath modes at the system transitions $\omega_n - \omega_m$ are in their vacuum states, so in Eq. (2) only terms of the form $\varphi\varphi^{\dagger}\rho_b$, $\varphi^{\dagger}\rho_b\varphi$, and $\rho_b\varphi\varphi^{\dagger}$ survive the trace over the bath. More simplification results because the φ_{nm} were assumed to be distinct bath modes for separate n and m. It then follows (see pp. 21–25 of Ref. [18]) that the explicit form for $-K^{\natural}\rho$ is

$$\sum_{n>m\geq 0} \frac{\gamma_{nm}}{2} (G_{nm}^{\dagger} G_{nm} \rho + \rho G_{nm} G_{nm}^{\dagger} - 2 G_{nm} \rho G_{nm}^{\dagger}),$$

where the spectral densities of the bath and the transition matrix elements of V are combined in

$$\gamma_{nm} = 2|\nu_{nm}|^2 \int_0^{\infty} \mathrm{tr}_b\{\varphi_{nm}\varphi_{nm}^{\dagger}(t)\rho_b\} e^{i(\omega_n - \omega_m)t}\, dt\,.$$

To find $R(t)$, note that L of Eq. (1) has only elements between $|\phi_n^i\rangle\langle\phi_n^j|$ and $|\phi_m^i\rangle\langle\phi_m^j|$ for any i and j, and for all $n > m \geq 0$. That includes coherences between the funnels, when $i \neq j$. In fact, these elements of L are independent of i and j, so L is block diagonal with blocks of the form (listed by $|\phi_n^i\rangle\langle\phi_n^j|$ with increasing $n = 0, 1, 2, \ldots$):

$$\begin{matrix} 0 & \gamma_{10} & \gamma_{20} & \gamma_{30}\cdots \\ 0 & -\gamma_{10} & \gamma_{21} & \gamma_{31}\cdots \\ 0 & 0 & -(\gamma_{20}+\gamma_{21}) & \gamma_{32}\cdots \\ 0 & 0 & 0 & -(\gamma_{30}+\gamma_{31}+\gamma_{32}) \end{matrix}\,.$$

This is in upper diagonal form, so the eigenvalues are on the diagonal. By Eq. (14), they all have a nonzero, negative real part, except for a single zero. The degeneracy of H_s was important here, because it prevented the separate coherences from evolving with distinct frequencies. Using the property that the elements in each column sum to zero, the right and left eigenvectors for the zero eigenvalue are $(1, 0, 0, \ldots)$ and $(1, 1, 1, \ldots)$, respectively. Thus,

$$R(t\to\infty) \to \sum_{i,j} |\phi_0^i\rangle\langle\phi_0^j| \otimes \left(\sum_{n>0} |\phi_n^i\rangle\langle\phi_n^j|\right). \tag{18}$$

The rate at which R approaches this limit is determined by $\exp(-\mu t)$, where $\mu = \min\{\mathrm{Re}\sum_{n>m}\gamma_{nm}\}$ is the eigenvalue with the least negative real part. To show that $R(\infty)$ is a recovery superoperator, consider a starting state of the system after an error:

$$\mathrm{tr}_e\{E\rho E^{\dagger}\} = \mathrm{tr}_e\left\{E\sum_{ij}\rho_{ij}|\phi_0^i\rangle\langle\phi_0^j|E^{\dagger}\right\}$$
$$= \sum_{a,b}\sum_{i,j}\sum_{n,m>0}\rho_{ij}\langle e_b\mid e_a\rangle\alpha_{an}\alpha_{bm}^{\star}|\phi_n^i\rangle\langle\phi_m^j|, \tag{19}$$

where the environmental states have been traced out. Applying $R(\infty)$, we can ignore $|\phi_n^i\rangle\langle\phi_m^j|$ with $n \neq m$ because they are not connected to P_0 (they decay to zero under R). After rearranging terms,

$$\xrightarrow{R} \left(\sum_{a,b}\sum_{n>0}\langle e_b\mid e_a\rangle\alpha_{an}\alpha_{bn}^{\star}\right)\rho\,. \tag{20}$$

In order that E conserve probability, it can be shown that the term in parentheses must be 1. The original ρ is thus recovered by evolution for a time $\mu t \gg 1$.

Finally, we mention that finding an optimum waiting period will depend upon factors such as the implementation of fault-tolerant manipulations, and is beyond the scope of this paper. Discussions of manipulating decoherence free subspaces can be found elsewhere [26].

A proposed test system.—To show that the criteria of Eqs. (8)–(15) are not too strict for any physically realizable example, we give one that implements Shor's three-qubit code against spin-flip errors [1]. Note that AQEC is not limited to spin-flip errors. This simple case serves only to illustrate that the AQEC criteria are not an impossible fiction.

Let three spin 1/2 particles be equally spaced along the z axis in zero static magnetic field. They interact by point dipolar $D_{nm}(I_{n,x}I_{m,x} + I_{n,y}I_{m,y} - 2I_{n,z}I_{m,z})$ and exchange $J_{nm}(I_{n,x}I_{m,x} + I_{n,y}I_{m,y} + I_{n,z}I_{m,z})$ terms [21]. We employ the Pauli operator formalism here. Dipolar interactions decrease with distance as r^{-3}, so $D_{12} = D_{23} = 8D_{13} = \zeta$, where ζ can be as large as 0.1 cm^{-1} [27]. If the system is dominated by dipolar interactions, then the level diagram is as in Fig. 1(a). The degenerate ground states $|000\rangle$ and $|111\rangle$ are chosen as the code words.

The degeneracy of H_s is due to conservation of angular momentum about z, $\sum_n I_{n,z}$ with eigenvalues m_z. The code words have $m_z = \pm 3/2$, and the funnel states $m_z = \pm 1/2$. Spin-flip errors $I_{n,x}$ take $m_z \to m_z \pm 1$, driving code words into their funnels and performing work on the system. Suppose the baths are photons with x-polarized B fields, so $V = (I_{1x} + I_{2x} + I_{3x})(a + a^{\dagger})$ [21]. Then the existence of an operator $8I_{1,x}I_{2,x}I_{3,x}$ that leaves V unaltered and maps $m_z \to -m_z$ implies equivalent dynamics between the two funnels.

There are some complications. Interaction with a y-polarized B field can antisymmetrically deexcite the funnels, altering the $I_{n,x}$ error to a $I_{n,z}$ error. Since rotating

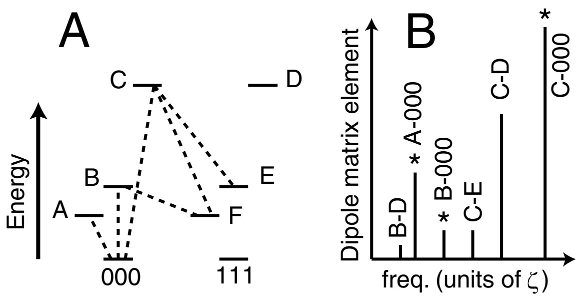

FIG. 1. (a) The level diagram of the three spin system. The code words are the ground states, and the two funnels are states $A-C$ and $D-F$. The dashed lines show the dipole-allowed transitions for an x-polarized B field from levels $A-C$. Identical transitions exist for $D-F$. (b) The magnitude of the matrix elements corresponding to the transitions. Only the starred transitions, which take the funnel states back down to their respective code words, are cooled.

a spin about z requires no work, this phase-flip error cannot be repaired and must be avoided. Next, dipolar interactions alone do not satisfy Eq. (14), because ν from B to A and $|000\rangle$ are both zero. Adding in the exchange interaction $0 < |J_{23}| \leq 0.5\zeta$ fixes this. For $J_{23} = 0.2\zeta$, the transition intensities of Fig. 1(b) result. The final difficulty is that besides the starred transitions (at 0.64ζ, 1.03ζ, and 2.39ζ), V drives funnel-funnel transitions as well. Emission at these frequencies must also be avoided. To prevent all these unwanted emissions, we employ a resonator. It can be shown [28] that placing the spins at the center of a rectangular resonator of linear dimensions $2.32/\zeta$, $0.87/\zeta$, and $4.28/\zeta$ produces modes with x-polarized B fields resonant with the starred transitions. There are other modes, but the nearest to an unwanted transition is offset from C-E by 0.018ζ. When the resonator $Q \gg 76$, emission here is effectively suppressed. If $\zeta \approx 0.1\ \text{cm}^{-1}$, a microwave resonator can easily achieve this goal. The resonator must be cooled to $T \ll (hc/k)\zeta \approx 0.1$ K to complete the system.

In conclusion, the conditions of Eqs. (8)–(15) imply that, in the weak coupling limit, the evolution superoperator of a system approaches the recovery superoperator against a given set of errors [9]. Thus, the ability to engineer the interaction between a bath and a system allows one to implement quantum error correction simply through dissipative dynamics, without requiring any external manipulation. An example that utilizes only well-understood magnetic interactions demonstrates the potential for a physically realizable implementation of AQEC.

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