Beam Envelope Equations for Cooling of Muons in Solenoid Fields

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Muon cooling is a critical component of the proposed muon collider and neutrino factory. Previous studies of cooling channels have tracked single muons through the channel, which requires many particles for good statistics and does not lend itself to an understanding of channel dynamics. In this paper, a system of moment equations are derived which captures the major aspects of cooling: interactions with material and acceleration by radio frequency (rf) cavities. A general analysis of solenoid lattice types compares well with prior simulations and indicates new directions for study.

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There has been significant research over the past five years on the feasibility of building a muon collider [1] and, more recently, growing excitement over the possibility of building a muon storage ring neutrino source [2,3]. Both of these concepts envision an intense proton beam incident on a target and producing pions, which then decay into muons. The production of muons is expensive and a substantial fraction of the initial six-dimensional muon phase space must be captured. Unfortunately, the initial muon phase space is orders of magnitude too large to be used in the collider or storage ring. The six-dimensional phase space volume must be reduced by a factor of order 100 before it can be useful for the neutrino factory, and 10^6 for the muon collider.

The cooling rates must be fast compared with the muon decay time. This restriction eliminates microwave stochastic cooling, used for antiprotons, and the heavy muon mass precludes synchrotron radiation damping, used in e^-e^+ machines. Indeed, the heavy mass of the muon is precisely why it is an attractive candidate for a high-energy collider: at 3 TeV, it can be bent in a circle with 7 km circumference and suffer insignificant radiation losses while, as a lepton, all its energy is available in the center of mass.

The cooling method that is believed most promising is ionization cooling [4], where particles are slowed down by passing through material and then accelerated with radio frequency (rf) cavities. Inside the material, particles lose momentum along their direction of motion, while the acceleration is strictly in the longitudinal direction. The result is a reduction in transverse angle. Competing against this transverse cooling are scattering events. The increase in beam phase space due to scattering is minimized by ensuring that the angular spread of velocities in the beam is as large as possible; this corresponds to focusing the beam to a small spot size. As the beam cools transversely, an unfavorable variation of energy loss with energy $d(dP/ds)/dP$ creates longitudinal heating. Eventually, particles can no longer be accelerated by the rf, and begin to be lost.

The method proposed to overcome this problem is emittance exchange, in which the six-dimensional phase space is manipulated so that some longitudinal phase space is added to the transverse space. The now larger transverse phase space can be reduced through ionization cooling. While there are various suggestions for implementing emittance exchange, none have been proven workable in any realistic situation. Present designs for the neutrino factory do not require emittance exchange—the phase space reduction of around 100 can be fully transverse. This paper studies the dynamics of transverse cooling and is restricted to azimuthally symmetric beams. Extensions to include nonaxisymmetric channels are straightforward and are likely to be needed for the study of emittance exchange as well as of beam lines with bends and magnet errors.

We first consider single particle motion in vacuum magnetic fields. The magnetic field inside of a cylindrically symmetric solenoid is $\vec{B} = \nabla \times [\mathcal{A}_\phi(r,z)\hat{e}_\phi]$. For a paraxial theory, it is sufficient to approximate $\mathcal{A}_\phi \simeq rB(z)/2$, where $B(z) \equiv B_z(r=0,z)$. The constants of motion are P^2 (or energy) and the canonical angular momentum, $L_{\rm canon} = xP_y - yP_x + qr\mathcal{A}_\phi$, where q is the charge. In a rotating coordinate frame (the Larmor frame), with $X_R = x\cos\varphi - y\sin\varphi$, $Y_R = x\sin\varphi + y\cos\varphi$, and

$$\varphi' = \frac{q\mathcal{A}_\phi}{P_z r} \simeq \frac{qB(z)}{2P_z} \equiv \kappa\,, \tag{1}$$

the linearized equations of motion reduce to $X_R'' = -\kappa^2 X_R$ and $Y_R'' = -\kappa^2 Y_R$; here, $'$ indicates the derivative along the z axis.

We can parametrize the solutions to the linearized equations in terms of a "betatron function," β_p, and phase, Φ, by $X_R = A_1\sqrt{\beta_p}\cos(\Phi - \Phi_1)$ and $Y_R = A_2\sqrt{\beta_p}\cos(\Phi - \Phi_2)$. The transverse amplitudes A_1 and A_2 correspond to the Courant-Snyder invariants [5]. The betatron function satisfies $\Phi' = 1/\beta_p$, and β_p evolves as

$$2\beta_p\beta_p'' - (\beta_p')^2 + 4\beta_p^2\kappa^2 - 4 = 0\,. \tag{2}$$

The angular momentum is, to lowest order, $L_{\rm canon} \simeq P_z A_1 A_2 \sin(\Phi_2 - \Phi_1)$. The forward momentum is then determined by $P^2 = P_z^2[1 + (x')^2 + (y')^2]$.

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Now consider a cylindrically symmetric distribution of particles in transverse phase space. Our goal is to find a simple description of the rms properties of the particle distribution. These are described by the matrix M of the second-order transverse beam moments. For simplicity, we neglect coupling between longitudinal and transverse moments. The matrix M, which is also the covariance matrix for the axis-centered beam, contains, by symmetry, only four independent moments in x, P_x, y, and P_y, and

$$\det M = [\langle x^2\rangle\langle P_x^2\rangle - \langle xP_x\rangle^2 - \langle xP_y\rangle^2]^2. \tag{3}$$

The transverse normalized rms emittance, ϵ_N, is defined as $\det M = m^4c^4\epsilon_N^4$. Because of the solenoid fields, M does not decompose into orthogonal planes.

The net canonical angular momentum is represented by a dimensionless parameter $\mathcal{L}$ through $\langle L_{\rm canon}\rangle \simeq \langle xP_y - yP_x + \kappa r^2 P_z\rangle = 2mc\epsilon_N\mathcal{L}$. The remaining quantities are characterized by the parameters $\alpha_\perp$, $\beta_\perp$, and $\gamma_\perp$, in analogy with the Courant-Snyder formalism. The symmetric moments matrix M is now written as

$$\frac{M}{mc\epsilon_N} = \begin{pmatrix} \beta_\perp/\langle P_z\rangle & & & \\ -\alpha_\perp & \langle P_z\rangle\gamma_\perp & & \\ 0 & \beta_\perp\kappa - \mathcal{L} & \beta_\perp/\langle P_z\rangle & \\ \mathcal{L} - \beta_\perp\kappa & 0 & -\alpha_\perp & \langle P_z\rangle\gamma_\perp \end{pmatrix}. \tag{4}$$

Here, the average value of P_z is used, and we set $\kappa = qB(z)/2\langle P_z\rangle$. The definition of emittance combined with Eq. (3) imposes the condition

$$\gamma_\perp \equiv \frac{1}{\beta_\perp}[1 + \alpha_\perp^2 + (\beta_\perp\kappa - \mathcal{L})^2]. \tag{5}$$

Starting from the deterministic single particle equations, $x' = P_x/P_z$, $y' = P_y/P_z$, and

$$v_z\frac{d\vec{P}}{dz} = \frac{d\vec{P}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) + \vec{v}\,\frac{dP}{ds}, \tag{6}$$

where $dP/ds < 0$ is the average change in momentum per path length in material, the equations of evolution can be found by interchanging the operations of averaging and differentiation so that, for example, $d\langle x^2\rangle/dz = \langle 2xP_x/P_z\rangle$. Then, to incorporate the effect of multiple scatter, an additional term P_zPS is added to the derivatives of $\langle P_x^2\rangle$ and $\langle P_y^2\rangle$; here, $S \equiv d\langle x'^2\rangle/ds \simeq (13.6\ \text{MeV}/vP)^2/L_R$, and L_R is the radiation length of the material, using a Gaussian fit [6] to the Moliere theory for multiple scatter.

Assuming that correlations between longitudinal and transverse quantities are weak (neglecting energy dispersion, for example), and considering in the transverse magnetic field only $B_r \propto r$, the evolution of the beam is determined by

$$\begin{aligned}
\epsilon_N' &= \beta_\perp\frac{PS}{2mc} + \epsilon_N\frac{1}{P_z}\frac{dP}{ds},\\
\beta_\perp' &= -2\alpha_\perp + \beta_\perp\frac{q\langle E_z\rangle}{v_zP_z} - \frac{\beta_\perp^2}{\epsilon_N}\frac{PS}{2mc}\\
&\quad - \frac{mc}{P_z}\beta_\perp\epsilon_N(\beta_\perp\kappa - \mathcal{L})\frac{qB'}{P_z},\\
\alpha_\perp' &= -\gamma_\perp + 2\kappa(\beta_\perp\kappa - \mathcal{L}) - \frac{\alpha_\perp\beta_\perp}{\epsilon_N}\frac{PS}{2mc},\\
\mathcal{L}' &= -\beta_\perp\kappa\frac{1}{P_z}\frac{dP}{ds} - \frac{\mathcal{L}\beta_\perp}{\epsilon_N}\frac{PS}{2mc},\\
\langle P_z\rangle' &= \frac{q\langle E_z\rangle}{v_z} + \frac{dP}{ds} - mc\epsilon_N(\beta_\perp\kappa - \mathcal{L})\frac{qB'}{P_z}.
\end{aligned} \tag{7}$$

This set of equations allows for interactions with material in addition to arbitrary changes in beam momentum. The equation for emittance growth is not new [7], but the coupled transport and cooling equations first appeared in [8]. A computer code for the evolution of moments based on symbolic manipulation of the Vlasov equation has been developed by Shadwick [9]. The formalism in this paper differs mainly by the development of an extension of the conventional beam dynamics parameters, which facilitates analytic study. In addition, analytic expressions for the beam cooling rates in terms of material properties are given in Ref. [10], for a fixed momentum beam in a lattice with a given beta function.

In a vacuum with only magnetic fields, the beam parameters evolve according to $\beta_\perp' = -2\alpha_\perp$ and

$$2\beta_\perp\beta_\perp'' - (\beta_\perp')^2 + 4\beta_\perp^2\kappa^2 - 4(1 + \mathcal{L}^2) = 0. \tag{8}$$

$\mathcal{L}$ and ϵ_N are constant, and $\gamma_\perp$ is given by Eq. (5). Note that $\beta_\perp = \beta_p\sqrt{1 + \mathcal{L}^2}$ relates the envelope beta function to the single particle β_p. Thus, canonical angular momentum makes beams harder to focus. In previous treatments [11], the envelope equation includes this contribution to the beam spot size by defining an "effective" emittance related to angular momentum; however, this is not useful in the context of ionization cooling, where only the *uncorrelated* spread of angles (i.e., true emittance) reduces the effect of multiple scattering.

This theory enables a straightforward analysis of a wide range of cooling channel geometries. It agrees well with a thin lens approximation, which is analytically tractable but not realistic for proposed muon cooling channels. Here, we study a more realistic model with extended solenoids. The field on axis is expanded in three Fourier harmonics in the form $B(z) = B_1\sin(2\pi z/L) + B_2\sin(4\pi z/L) + B_3\sin(6\pi z/L)$, where L is the periodicity of the magnetic field. For a given momentum P_z and charge $|q| = e$, the lattice is characterized by the relative sizes of B_1, B_2, and B_3, together with the quantity

$$\chi \equiv \frac{B_{\max}[\mathrm{T}]L[\mathrm{m}]}{P_z\ [\mathrm{GeV/c}]} \simeq 6.67\kappa_{\max}L, \tag{9}$$

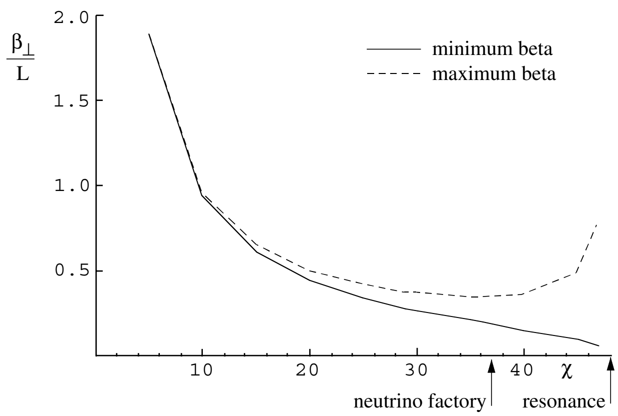

FIG. 1. Analytic calculations of $\beta_{\min}/L$ (solid line) and $\beta_{\max}/L$ (dashed line). The parameters for a proposed FOFO cooling channel for a neutrino source is indicated.

where $B_{\max}$ is the maximum amplitude of the magnetic field on axis. The transport properties of channels without acceleration or material can be described by the momentum acceptance $\Delta P/P$, defined as the minimum relative momentum shift which will put the beam into unstable transverse motion, and $\beta_\perp/L$.

We first consider the so-called FOFO ("focusing-focusing") lattice where the magnetic field on axis is a simple sinusoid ($B_2 = B_3 = 0$). The momentum acceptance of a FOFO channel has been examined in detail by Fernow [12], starting from the equation for the beam size, $R \propto \sqrt{\beta}$, which in a vacuum with no electric fields is determined by $R'' + \kappa^2 R - m^2c^2\epsilon_N^2(1 + \mathcal{L}^2)/R^3P_z^2 = 0$. The unstable regions were found by neglecting the $1/R^3$ term, and looking for unbounded solutions of the resulting Mathieu equation. These results agree with numerical calculations of periodic solutions to Eq. (8). For the FOFO lattice, all beams with $\chi < 48.0$ can be propagated by the channel; this corresponds to all momenta above some resonant value, at which the phase advance per half period is 180 degrees. Immediately below this momentum, particles undergo unstable motion in the transverse direction; however, there are other momentum intervals which support stable motion.

In addition to determining the momentum acceptance, the envelope equations give detailed information about the beta function. At low χ, $\beta_\perp(z)$ is roughly constant at $\simeq 9.4L/\chi$; the focusing is determined by the average of $B^2(z)$. The minimum and maximum of the beta function, which, respectively, determine the cooling potential of and the aperture required for a given beam, are shown in Fig. 1 for $\chi < 48.0$; a rough numerical fit is

$$\beta_{\substack{\min\\ \max}} \simeq L\,\frac{9.4}{\chi}\left[1 - \left(\frac{\chi}{48.0}\right)^2\right]^{\pm 1/2}. \qquad (10)$$

Close to resonance, this implies that the minimum beta scales as $\beta_{\min} \simeq 0.28L(\Delta P/P)^{1/2}$, where $\Delta P/P = 1 - (\chi/48.0)$. Thus, although the minimum of the beta function can be reduced by taking the beam momentum closer to resonance, the cost in terms of the reduction to the momentum acceptance is high. The maximum of the beta function reaches its smallest value of $\simeq 0.32L$, at $\chi \simeq 34$. The technological difficulty in achieving a short periodicity L limits the cooling performance of FOFO channels.

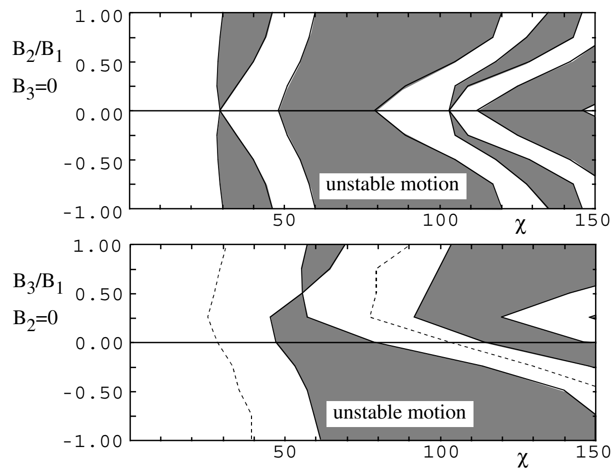

FIG. 2. Phase diagrams for lattice behavior when a harmonic is added to a sinusoidal magnetic field. The stability for a given second (top) and third (bottom) Fourier component is indicated as a function of $\chi \propto 1/P_z$. For the third harmonic, half-integer tunes are indicated (dashed lines).

In Figure 2, "phase diagrams" delineating the boundaries between bounded and unbounded motion are shown for the two sets of cases where $B_3 = 0$ or $B_2 = 0$. The FOFO geometry corresponds to $B_2 = B_3 = 0$. Other previously described cooling channel configurations [13] are analogous to various positions in the second interval of stable motion. Note that for $B_2 = 0$, $B_3/B_1 \simeq 0.5$, the first region of unstable motion becomes extremely small.

In the center of the second region of stability, the minimum of the beta function typically scales as $(\Delta P/P)^2$ when the geometry of the lattice is altered. In contrast, when the momentum is shifted closer to resonance, $\beta_{\min} \propto (\Delta P/P)^{1/2}$, as in the case for the FOFO lattice. Thus, it is most efficient to center the region of momentum accep-

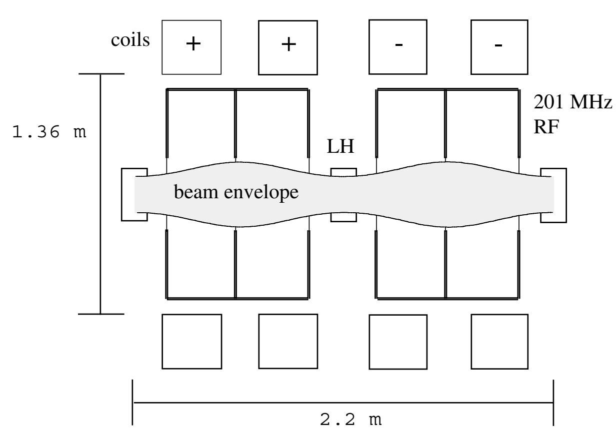

FIG. 3. FOFO cooling lattice: diagram of coils, rf cavities, liquid hydrogen (LH) chambers, and beam envelope.

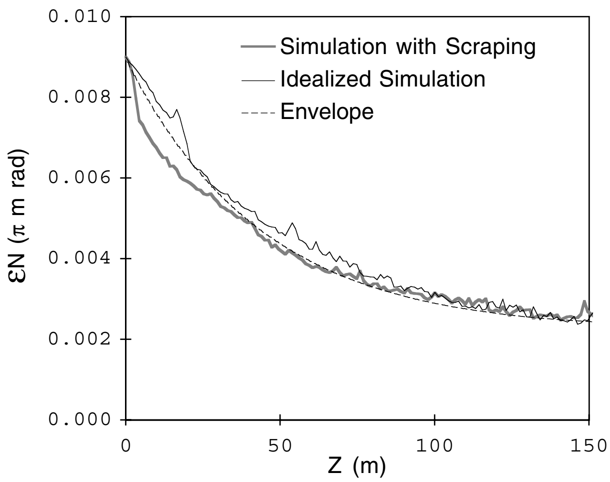

FIG. 4. FOFO cooling lattice: Transverse emittance from envelope equations (dashed line) compared with 1000 particle ICOOL simulation with (thick line) and without (thin line) realistic apertures.

tance about the target momentum, and to achieve small $\beta_{\min}$ by varying the geometry.

Solutions to the moment equations yield good agreement with single particle tracking codes, such as ICOOL [14]. As an example, consider a FOFO cooling channel using liquid hydrogen vessels that has been proposed for a neutrino factory. The magnetic field period is 2.2 m, the peak magnetic field on axis is 3.4 T, and the beam momentum is 0.2 GeV/c. The geometry is indicated in Fig. 3 and the transverse emittances from simulations are shown in Fig. 4. There is close agreement between the moment equations and simulation results, although in the simulation which incorporates realistic apertures for the beam line there is an initial sharp drop in emittance due to beam scraping.

To illustrate the effect of canonical angular momentum, we consider a 0.2 GeV/c beam propagating in a uniform 5 T solenoid. Liquid hydrogen vessels and rf are arranged in a 1.1 m period lattice. After 88 meters, there is a sharp field reversal into a -5 T solenoid. The resultant transverse emittance and canonical momentum are shown in Fig. 5. The results of an ICOOL particle simulation are also shown. Without the field flip to reverse the buildup of angular momentum, the beam emittance would saturate at a significantly higher value.

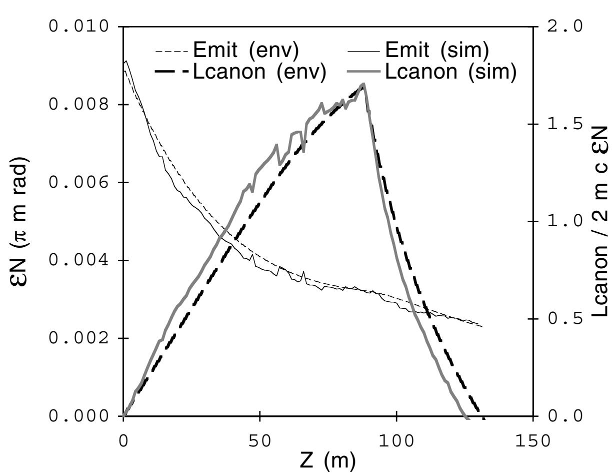

FIG. 5. Predictions for cooling in a solenoid with field reversal after 88 m, from envelope equations (dashed lines) and ICOOL simulation (solid lines). Transverse emittance (thin lines) and scaled canonical momentum (thick lines) are shown.

A paraxial theory for beam moment equations in solenoid fields has been developed, and applied towards lattices designed for ionization cooling of muons. This theory is similar in form to the Courant-Snyder formalism for quadrupole focusing systems, and allows for a rapid analysis of cooling channel performance, including the development of scaling laws. These results are consistent with particle tracking codes and require a small fraction of the computational cost. Extensions of this theory to the full six-dimensional phase space will allow treatment of beam asymmetries, bending magnets, error analysis, nonlinear fields, and space-charge effects.

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