Computation of the Vortex Free Energy in SU(2) Gauge Theory

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We present the first measurement of the vortex free-energy order parameter at weak coupling for SU(2) in simulations employing multihistogram methods. The result shows that the excitation probability for a sufficiently thick vortex in the vacuum tends to unity. This is rigorously known to provide a necessary and sufficient condition for maintaining confinement at weak coupling in SU(N) gauge theories.

The vortex free-energy (also known as magnetic-flux free-energy) order parameter in gauge theories is defined as the ratio of the partition function in the presence of a topologically trapped vortex excitation (introduced by a singular gauge transformation) to that without it. Its Fourier transform with respect to the center $[Z(N)]$ of the gauge group $[SU(N)]$ defines the so-called electric-flux free energy which is rigorously known to provide an upper bound on the Wilson loop. These flux order parameters can characterize all possible phases of a (pure) gauge theory, and furthermore do this in terms of the behavior of the excitation expectation for a vortex. They were first considered in the study of gauge theories in [1], though the use of the analogous quantities in statistical mechanics goes back much further [2]. The idea that vortex configurations underlie confinement at weak coupling has a long history, and has been the subject of intense recent activity. (We refer to Ref. [3] for a review of recent developments and references to early and recent work.)

In view of the physical significance of the magnetic-flux free energy, it may appear surprising that it has not been measured in simulations over the last 20 years. Accurate determination of (differences of) free energies in gauge theories, however, is well known to be difficult. In fact, it is at first not quite clear how one should go about computing such totally nonlocal (lattice-length) quantities. We present here a computation for the group SU(2) based on multihistogram methods [4]. Such a method was recently used in Ref. [5] to compute the free energy of a pair of $Z(N)$ monopoles, a quantity related to the 't Hooft loop operator. Our result demonstrates that the excitation expectation for a sufficiently extended "thick" vortex at large β is essentially unity. This is the feature responsible for maintaining the confining phase in SU(N) gauge theories even at weak coupling.

We work on a d-dimensional hypercubic lattice Λ of size $L_1 \times \cdots \times L_d$ with periodic boundary conditions in all directions. We generally denote bonds by b, plaquettes by p, cubes by c, etc. The plaquette action is denoted by $A_p(U_p)$, where, as usual, $U_p = \prod_{b \in p} U_b$, the product of the bond variables around the plaquette; for the minimal

(Wilson) action $A_p(U_p) = -\beta \operatorname{Re} \operatorname{tr} U_p$. The trace "tr" is defined to include a $1/N$ normalization.

A coclosed set of plaquettes (2 cells) is a closed set of $(d-2)$ cells on the dual lattice. Thus, in $d = 3$, it is a closed loop of dual bonds; in $d = 4$, it is a closed two-dimensional surface of dual plaquettes. For fixed μ, ν, let $\mathcal{V}_{\mu\nu}$ denote a coclosed set of plaquettes that winds through every two-dimensional $[\mu\nu]$ plane of Λ, i.e., a topologically nontrivial plaquette set wrapped around the periodic lattice (d torus) in the $(d-2)$ directions $\lambda \neq \mu, \nu$ perpendicular to μ, ν. This is depicted in Fig. 1(a), where the short lines represent the plaquettes in $\mathcal{V}$, with the horizontal axis representing the x^μ, x^ν directions, and the vertical axis the remaining $(d-2)$ perpendicular directions.

Define the partition function

$$Z_\Lambda(\tau_{\mu\nu}) = \int \prod_b dU_b \exp\Bigg(- \sum_{p \notin \mathcal{V}_{\mu\nu}} A_p(U_p) - \sum_{p \in \mathcal{V}_{\mu\nu}} A_p(\tau_{\mu\nu} U_p) \Bigg), \quad (1)$$

where the plaquette action $A_p(U_p)$ is replaced by the "twisted" action $A_p(\tau_{\mu\nu} U_p)$ for each plaquette of $\mathcal{V}_{\mu\nu}$. Here the "twist" $\tau_{\mu\nu} \in Z(N)$ is an element of the center.
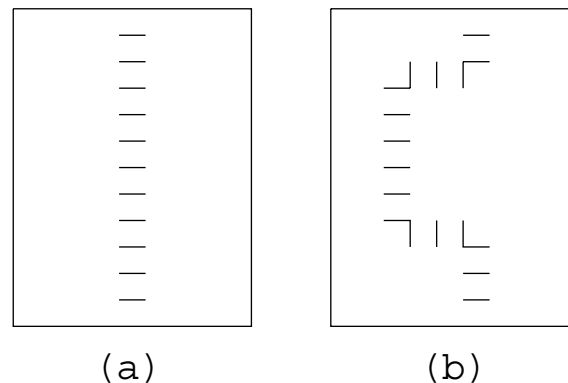

(a)   (b)

FIG. 1. Stack of plaquettes carrying twist winding around the periodic lattice. (a) and (b) are equivalent sets.

There are thus $(N - 1)$ different nontrivial choices for $\tau_{\mu\nu}$. The trivial element $\tau_{\mu\nu} = 1$ is the ordinary partition function $Z_{\Lambda}(1) \equiv Z_{\Lambda}$.

As indicated by the notation on the left hand side of (1), the exact position or shape of $\mathcal{V}_{\mu\nu}$ is irrelevant; the only dependence is on the presence of the $Z(N)$ flux winding through each $[\mu\nu]$ plane. It is indeed easily seen that $\mathcal{V}_{\mu\nu}$ can be moved around and distorted by a shift of integration variables, but not removed; it is rendered topologically stable by winding completely around the lattice [Fig. 1(b)]. By the same token, introducing two twists, $\tau_{\mu\nu}$ on $\mathcal{V}_{\mu\nu}$ and $\tau'_{\mu\nu}$ on $\mathcal{V}'_{\mu\nu}$ in (1), is equivalent to introducing one twist $\tau''_{\mu\nu} = \tau_{\mu\nu}\tau'_{\mu\nu}$ since $\mathcal{V}_{\mu\nu}$ and $\mathcal{V}'_{\mu\nu}$ can be brought together by a shift of integration variables (Fig. 2). This expresses the mod$N$ conservation of the $Z(N)$ flux introduced by the twist. Thus, for $N = 2$, any odd number of such (nontrivial) twists is equivalent to one, and any even number to none.

The magnetic-flux free energy order parameter is now defined as

$$\exp[-F_{\text{mg}}(\tau_{\mu\nu})] = \frac{Z_{\Lambda}(\tau_{\mu\nu})}{Z_{\Lambda}}$$

$$= \left\langle \exp\left(- \sum_{p \in \mathcal{V}_{\mu\nu}} [A_p(\tau_{\mu\nu}U_p) \right. \right.$$

$$\left. \left. - A_p(U_p)] \right) \right\rangle. \quad (2)$$

Generalizations of (1) and (2) may be considered by introducing sets $\mathcal{V}_{\kappa\lambda}$ for several or all of the $\frac{1}{2}d(d-1)$ possible *distinct* choices of planes $[\kappa\lambda]$.

The twist amounts to a discontinuous (singular) SU(N) gauge transformation on the configurations in (1) with multivaluedness in $Z(N)$ [so it is single valued in SU(N)/$Z(N)$], i.e., the introduction of a $\pi_1[\text{SU}(N)/Z(N)] = Z(N)$ vortex. The set $\mathcal{V}_{\mu\nu}$ represents the topological obstruction to having single valuedness everywhere. Equation (1) is then the partition sum for the system with a topologically stable vortex completely winding around the lattice, and Eq. (2) is the normalized expectation for
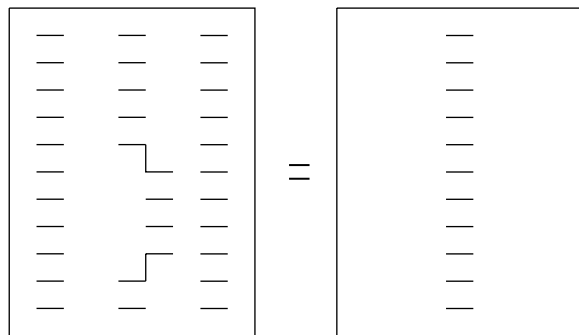

FIG. 2. Equivalent sets $\mathcal{V}$ reflecting mod$N$ conservation of the twist ($N = 2$).

the excitation of such a vortex. Hence, it is also referred to as the vortex free energy.

Choosing, say, $[\mu\nu] = [12]$ in (1), we now drop the $\mu\nu$ subscript. One is interested in the behavior of (2) in the large volume limit (in the van Hove sense), i.e., as the size of the lattice increases in any power law fashion, e.g., $L_{\mu} = 2^{la_{\mu}}$ for some fixed choice of positive exponents a_{μ}, integer $l \to \infty$. Let $A = L_1 L_2$ be the area of each $[12]$ plane, and $L = L_3 \ldots L_d$ the lattice volume in the perpendicular directions. One is interested, in particular, in $L \geq A$. The twist introduces a cost in action localized on the plaquettes in $\mathcal{V}$. This cost, proportional to L, may be lowered if there are configurations that contribute with finite measure in the integral (2), and allow the flux introduced by the twist to spread in the two directions perpendicular to $\mathcal{V}$, so that the action is closer to its minimum, in other words, if there is finite probability for exciting a "thick" vortex.

For sufficiently large lattices, there are then the following three possibilities ($\tau \neq 1$):

(a) $\exp[-F_{\text{mg}}(\tau)] \sim \exp[-\alpha(\beta, \tau)L]$,

(b) $\exp[-F_{\text{mg}}(\tau)] \sim \exp\left(-\beta c(\tau)\frac{L}{A}\right)$,

(c) $\exp[-F_{\text{mg}}(\tau)] \sim \exp(-cLe^{-\rho(\beta,\tau)A})$.

In case (a) the magnetic flux stays focused in a thin vortex; this describes a Higgs phase. In (b) the flux can spread in a Coulomb-like fashion lowering the free-energy cost; this describes a massless Coulomb phase, where the long distance behavior is accurately given by weak coupling perturbative expansion. In (c) the gain in thickening the vortex is exponential; this characterizes the confinement phase. It is important to note that, in contrast to (a) and (b), only (c) gives a value which survives and, in fact, tends (exponentially) to unity for all ways of taking the thermodynamic limit as described above; this is the signature of the confinement phase.

Since our computation below is for $N = 2$, we now write explicit formulas only for this case. The Fourier transform of (2) with respect to $Z(N)$ is known as the electric-flux free energy. For $N = 2$ this is simply

$$\exp(-F_{\text{el}}) = \sum_{\tau=1,-1} \tau \exp[-F_{\text{mg}}(\tau)]$$

$$= 1 - \exp[-F_{\text{mg}}(-1)]. \quad (3)$$

Consider now a rectangular loop C in a $[12]$ plane. Then, for any reflection positive plaquette action, the Wilson loop obeys the bound [6]

$$\langle \text{tr}(U[C])\rangle \leq [\exp(-F_{\text{el}})]^{A_C/A}, \quad (4)$$

where A_C is the minimal area bounded by C. Equation (4) shows that confining behavior (c) for the vortex free energy implies area law for the Wilson loop with string tension bounded from below by the excitation expectation for

a vortex. So confining behavior for the vortex free energy is a *sufficient condition* for linear asymptotic quark confinement.

Placing suitable constraints in the functional measure in (1) which forbid the spreading of flux across [12] planes, thus eliminating the occurrence of thick vortices, results in the nonconfining behavior of type (a) above [7]. In this case (4) cannot tell us anything about the Wilson loop. To show loss of confining behavior for the Wilson loop itself in the presence of such constraints, one needs a *lower* bound on it which exhibits perimeter law. This was recently proven for large β in [8]. Thus the occurrence of thick vortices is also a *necessary condition* for confinement at weak coupling.

Our measurement of (2) for SU(2) was done by an application of a multihistogram method [4]. From now on we restrict the form of the action to the Wilson action

$$A_p(U_p) = -\beta \operatorname{tr} U_p, \qquad (5)$$

which was used in the measurement. The basic quantity in our procedure is the density of states $w(S)$ as a function of the total action S along the twisted plaquettes. This is defined as

$$w(S) = \prod \int dU_b \exp\left(\beta \sum_{p \notin \mathcal{V}} \operatorname{tr} U_p\right) \delta\left(S + \sum_{p \in \mathcal{V}} \operatorname{tr} U_p\right). \qquad (6)$$

If $w(S)$ is known, the partition function can easily be computed for any coupling $\beta_{\mathcal{V}}$ along $\mathcal{V}$ as

$$Z(\beta_{\mathcal{V}}) = \int dS\, w(S) e^{-\beta_{\mathcal{V}} S}. \qquad (7)$$

In particular, we are interested in $Z(\beta)$, the untwisted, and $Z(-\beta)$, the twisted partition function. The problem is that the dominant contribution for $Z(\beta_{\mathcal{V}})$ comes from different regions of S, depending on $\beta_{\mathcal{V}}$. Therefore one needs to know $w(S)$ to a good accuracy in a wide range of S. A simulation done at a certain value of $\beta_{\mathcal{V}}$, however, will give accurate information on $w(S)$ only in a narrow neighborhood of $\langle S \rangle_{\beta_{\mathcal{V}}}$. The main idea of the Ferrenberg-Swendsen multihistogram method is to combine information on $w(S)$ coming from simulations at different $\beta_{\mathcal{V}}$'s to obtain $w(S)$ in a wide range of S accurately. This can be done by noting that for a given $\beta_{\mathcal{V}}$ the probability distribution of S, $P(S, \beta_{\mathcal{V}})$, goes as

$$P(S, \beta_{\mathcal{V}}) \propto \frac{1}{Z(\beta_{\mathcal{V}})} e^{-\beta_{\mathcal{V}} S}, \qquad (8)$$

and that $P(S, \beta_{\mathcal{V}})$ can be directly measured by making a histogram of the action along $\mathcal{V}$. In this way, any simulation at a certain $\beta_{\mathcal{V}}$ gives an estimate for $w(S)$,

$$w(S) = P(S, \beta_{\mathcal{V}}) e^{\beta_{\mathcal{V}} S} Z(\beta_{\mathcal{V}}). \qquad (9)$$

These estimates coming from simulations with different $\beta_{\mathcal{V}}$'s (say $\beta_1, \beta_2, \dots, \beta_K$) can then be averaged with suitable (S dependent) weights to minimize the error in $w(S)$

over a given range of S. This results in the following set of coupled equations:

$$w(S) = \frac{\sum_{n=1}^{K} P(S, \beta_n)}{\sum_{n=1}^{K} \frac{\exp(-\beta_n S)}{Z(\beta_n)}}, \qquad (10)$$

$$Z(\beta_n) = \int dS\, e^{-\beta_n S} w(S), \qquad (11)$$

which can be solved by iteration starting from $Z(\beta_n) = 1$. To optimize the procedure, one needs a sufficient overlap between the $P(S, \beta_n)$ distributions corresponding to successive β_n's. Since the distributions quickly become narrower with increasing lattice size, the number of simulations, K, also needs to be increased accordingly. This makes our measurement very expensive on large lattices. For the largest lattices we typically used $K = 40$–80 simulations with the β_n's equally spaced in the $-\beta + \beta$ range.

The result of the computation for (2) is shown in Fig. 3. We have performed the computation on lattices of equal linear size in all directions for three different values of β. The lattice spacings are $a = 0.165$ fm, $a = 0.119$ fm, and $a = 0.085$ fm for $\beta = 2.3$, $\beta = 2.4$, and $\beta = 2.5$, respectively.

Notice that, with the lattice size expressed in physical units, the measurements for different β's fall on the same curve, as they should. This indicates that the universal curve has been reached, and will not change at larger beta. Also, the onset of the sharp rise around 0.7 fm is in the region of the finite temperature deconfining phase transition providing another indirect consistency check.

The approach to unity for sufficiently large lattice size in Fig. 3 is striking. In comparison, for Coulomb-like massless behavior, an upper bound obtained by action minimizing within the spin-wave approximation gives $\sim \exp[-\beta(\pi/2)^2] \approx 0.085$ at $\beta = 2.3$. The points forming the upper part of the plot are well within the
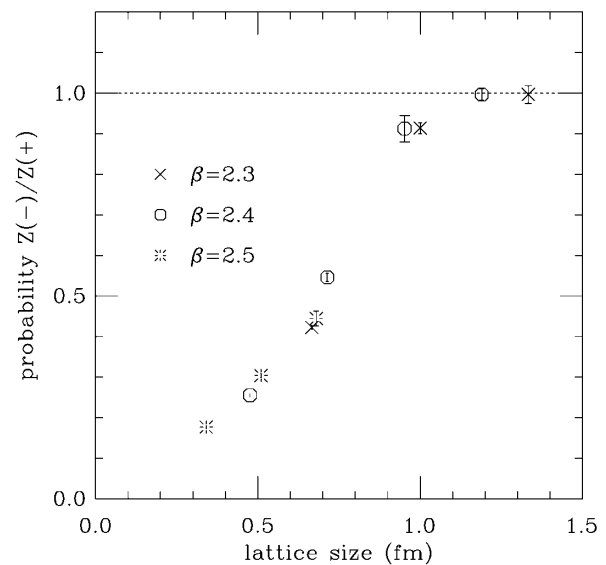

FIG. 3. SU(2) vortex free energy (2) vs lattice size.

confinement region. The string tension values extracted from the vortex free energy in the confining region are consistent with the values from heavy-quark potential calculations (see e.g., [9]), though still better precision in the measurement of the vortex free energy is required for precise quantitative comparisons.

In conclusion, the result of our computation clearly demonstrates that the weighted expectation for the excitation of a sufficiently thick vortex in the vacuum tends to one. In this sense the vacuum can indeed be viewed as having a "condensate" of thick long vortices. This is sufficient for maintaining confinement at large β in SU(N) gauge theories. As mentioned, rigorous results also show it to be necessary: were the behavior for (2) exhibited in Fig. 3 not to occur, confinement at large beta would be lost.

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