Invariant Integral and the Transition to Steady States in Separable Dynamical Systems

G. L. Lippi,* S. Barland, and F. Monsieur†
Institut Non Linéaire de Nice, UMR 6618 CNRS-Université de Nice-Sophia Antipolis,
1361 Route des Lucioles, F-06560 Valbonne, France
(Received 15 October 1999; revised manuscript received 14 February 2000)

We show that the transition between fixed points in a separable dynamical system is fully described by an invariant integral. We discuss in detail the case of a system with two temporal variables with bilinear coupling, where the new stable state is attained asymptotically through spiraling into the fixed point. Through the invariance, it is possible to establish conditions for the control parameter that permit a (targeted) transition in finite time and without relaxation oscillations.

PACS numbers: 05.45.–a, 42.55.–f, 42.60.Mi

Many aspects of the temporal evolution of dynamical systems have been studied in the past years and numerous techniques have been devised to stabilize unstable states (e.g., periodic orbits) embedded in a chaotic regime [1]. Such schemes are typically based on the knowledge of the local structure of the phase space and on the measurement of the instantaneous state of the system. Corrections to the control parameter(s) are issued on the basis of the deviation between the desired evolution and the actual one.

We present a *global steering technique* (global targeting) which permits the modification (and *optimization*) of the transition between two states and is based solely on the knowledge of the system's flow. Our scheme exploits the *global flow* and the associated changes coming from *macroscopic* variations in a control parameter. We therefore steer the system in phase space through an "artificial trajectory" which is the result of the transit through an infinite number of the system's own manifolds. The transit is driven by the variations—transverse to the manifolds—that we impose to the control parameter. In this way, no measurement of the instantaneous state of the system is necessary and no feedback need be applied. Furthermore, the scheme applies to all dynamical systems whose dynamics are *separable* in one variable and which have *one stable fixed point*.

For the sake of concreteness and without loss of generality, we discuss the principle of the *invariant integral* and present an example of the application of this technique to a specific type of nonlinear system which satisfies the previous requirements, consists of two dynamical variables, and possesses a bilinear nonlinearity. Numerous systems are well described by this type of model (e.g., population problems [2,3] and lasers [4]).

We consider the simple, separable dynamical system:

$$\dot{x} = -x + xy\,, \tag{1a}$$

$$\dot{y} = -\epsilon(y + xy - P)\,, \tag{1b}$$

where the dot represents the (rescaled) time derivative, x and y are the variables, ϵ ($\ll 1$) is the relaxation constant of y (relative to that of x to which time is rescaled), and P is the control parameter for the system. This model provides a good qualitative description of so-called class B lasers (solid-state, CO_2, semiconductors) [4] but could also describe population dynamics [2,3]. At variance with the classic Lotka-Volterra model [2,3], the population system that we consider here does not have an infinite amount of food supply for the prey (y), but rather a limited one, controlled from the "outside" through the parameter P. We also introduce a natural death rate for the prey, $-\epsilon y$, and the bilinear term represents the disappearence of prey due to predators (x) in Eq. (1b) [the same term is the source for predators, cf. Eq. (1a)]. Finally, as in [2,3], a death rate for the predator is introduced [$-x$ in Eq. (1a)]. We can therefore think of this system as a "cage" in which the prey is fed from the outside at a given rate and where the number of predators is related to the prey availability. The modifications that we introduce in this model with respect to the classic Lotka-Volterra model [2,3] drastically change the phase space structure, and interesting new features appear (cf. the discussion below and compare, e.g., to the analysis in [2,3]). For the laser, x and y represent the electromagnetic field intensity and the population inversion, respectively, and P represents the pump parameter. A more detailed interpretation of the model for this problem can be found in [5,6]. Throughout the paper we will consider $\epsilon > 0$ and $P > 0$ for obvious physical reasons, although the mathematical discussion is valid outside these bounds.

The properties of the steady states, their stability, and local phase space structure are trivially obtained from Eqs. (1a) and (1b) and are summarized in Fig. 1. In the whole parameter space there exists always one, and only one, stable fixed point with an exchange of stability when the control parameter passes through its "critical" 1 value. We are interested in the transition from **A** to **B** (Fig. 1) which brings the system from the trivial state (laser field intensity or predator population equal to zero) to the nontrivial state (laser field intensity or predator population different from zero); i.e., we are interested in describing the growth of the variable that "feeds off" the other one. The phenomenon we analyze is fully deterministic and therefore our treatment does not require the inclusion of fluctuations.

 0031-9007/00/85(1)/62(4)$15.00

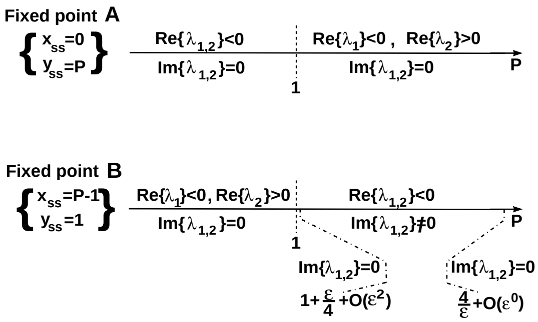

FIG. 1. Fixed points and their stability in parameter space for Eqs. (1a) and (1b). Point **A** is stable for $P \le 1$, point **B** above $P = 1$. We remark that when stable **B** is a focus on most of the parameter interval, since $\epsilon \ll 1$. Thus, the center appears either in a negligibly small interval $1 \le P \le 1 + \frac{\epsilon}{4}$ or for P values that are unrealistically high ($P > \frac{4}{\epsilon}$).

Formal integration of Eq. (1a) provides the condition $\log[x(t')] - \log[x(t_0)] = \int_{t_0}^{t'} [y(t) - 1]\,dt$. Since, as shown in the previous discussion, $x(t')$ and $y(t')$ always converge towards the fixed point **B** (for $P > 1$), then

$$\lim_{t' \to \infty} \log\left(\frac{x(t')}{x_0}\right) = \lim_{t' \to \infty} \int_0^{t'} [y(t) - 1]\,dt\,, \qquad (2)$$

where $x(t' \to \infty)$ and x_0 are the asymptotic and initial values of x [7], respectively, and t represents the integration time (over the semi-infinite interval, starting from the initial time chosen to be $t_0 = 0$). This condition relates the asymptotic value of x to its initial value through the integral of the deviation of y from its asymptotic value ($y_{ss,\mathbf{B}} = 1$) and we can interpret it as the "energy" that has to be provided to bring the system onto **B**. We remark that the existence and invariance of the integral [right-hand side (rhs) of Eq. (2)] are both nontrival and widely applicable [8].

Since $\epsilon \ll 1$, **B** is a focus over most of its stable P interval (Fig. 1) [9]. Hence, the approach to equilibrium will be achieved with oscillations [6], i.e., x and y exchange energy until they both reach *asymptotically* their respective values $x(t' \to \infty) = x_{ss,\mathbf{B}}$, $y(t' \to \infty) = y_{ss,\mathbf{B}}$. Only then is the new equilibrium state attained.

Figure 2 shows the evolution of the trajectory (solid line) from the initial point (**A** turned unstable) towards the final state (**B**) plotted over the phase space flow (vectors). The trajectory first evolves near the y axis, and only after passing $y = 1$ is it then shot up by the flow towards large values of x. Because of the focuslike structure of **B**, the trajectory is forced to roll itself around the fixed point.

The winding of the trajectory around **B** illustrates the fact that during the temporal evolution the system "exchanges" deficits and overabundance in either variable (Fig. 3a) until, gradually, the oscillating integral converges to the result of Eq. (2). Figure 3b shows the value taken by the rhs of Eq. (2) as a function of time for different values of y_0. In spite of the fact that the integral itself changes in time, all asymptotic values are equal. This numerically confirms the fact that the steady state is attained *only* by

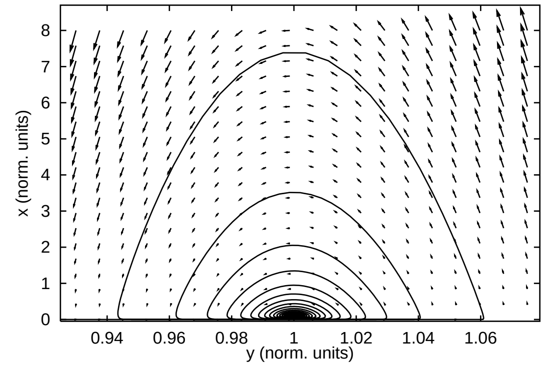

FIG. 2. Trajectory demonstrating the evolution from an intially prepared state (near **A**, unstable) into **B**. The flow is plotted in the background (vectors). The repeated winding of the trajectory follows the flow and forces the system through a long evolution to reach **B**. The transition results from suddenly setting the control parameter to the value for which **B** is stable at time $t = 0$. Parameter values (here and in the following figures, unless otherwise indicated): $\epsilon = 0.0025$, $x_0 = 1 \times 10^{-15}$, $y_0 = 0.9$, and $P_\infty = 1.1$.

waiting an infinite amount of time and that the integral, Eq. (2), is an *invariant*.

The question arises whether it is possible to obtain a direct transition from **A** to **B** and in finite time. Figure 2 shows that this is impossible under stepwise switching, since the trajectory cannot cross itself to enter **B** directly. However, there is a way of achieving the desired goal.

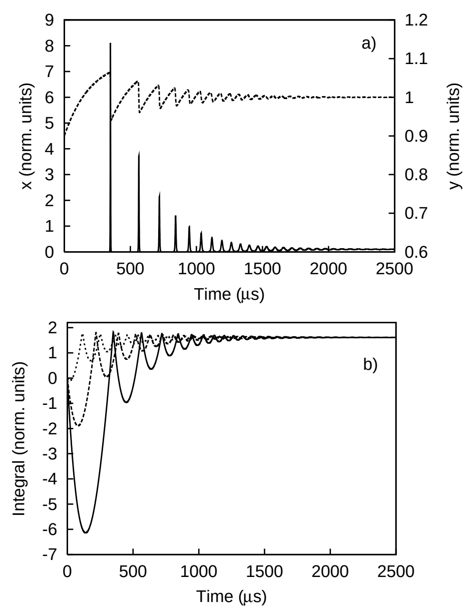

FIG. 3. (a) Temporal evolution of x (solid line) and y (dashed line) as a function of time for the same parameters as in Fig. 2. (b) Temporal evolution of the integral, Eq. (2), as a function of time for the same conditions as in Fig. 2 (solid line); for $y_0 = 0.95$ (long-dashed line); for $y_0 = 0.99$ (short-dashed line). Notice that the integrals converge *at different times* all to the *same value*. The integrals converge to $1.612 = \frac{1}{20}\log(\frac{x_{ss,\mathbf{B}}}{x_0})$, since time is in μs and the x rescaling constant we use is 2×10^7 s^{-1}.

The asymptotic result, Eq. (2), shows that the new equilibrium position (**B**) will be attained *when* its left-hand side (lhs) and rhs are equal *simultaneously*. This condition is certainly satisfied *asymptotically,* but if we find a way of satisfying it at a finite time $\bar{t}$, then the long oscillations seen in Fig. 3 will disappear. This is possible if at the *first* instant $t = \bar{t}$ at which $x(\bar{t}) = x_{ss,\mathbf{B}}$ the integral is satisfied [10]:

$$\log\left(\frac{x_{(ss,\mathbf{B})}}{x_0}\right) = \int_0^{\bar{t}} [y(t) - 1]\, dt\,, \tag{3}$$

and, in addition, $y(\bar{t}) = 1$ [equivalent to imposing that the time derivative of the rhs of Eq. (3) be zero at $t = \bar{t}$]. Such requirements can be translated into conditions to be imposed onto the control parameter P.

In other words, in order to attain this goal the system must acquire some additional degree of freedom which permits the appropriate modification of the integrand. The geometry of the phase space suggests that this may be possible by allowing the trajectory to exit the x-y plane. Indeed, while the trajectory *must* spiral into the *fixed point,* it can go straight into many other points of phase space (e.g., the point with coordinates $y = 1$ and maximum x on the trajectory in Fig. 2). If, at the correct time $\bar{t}$ [the one for which $x(\bar{t}) = x_{ss,\mathbf{B}}$], P is suddenly changed, then we may bring the representative point onto the final value and avoid the spiraling. This explains the technique used in [11]. However, use of the invariant integral allows us to obtain much better results than those of [11]. Indeed, we can use Eq. (3) as a criterion for steering the trajectory around in phase space and tailor the evolution to our desires. In the general form, the integral [Eq. (3)] is implicit and can best be embedded into a numerical optimization technique (since the system is nonlinear). Nevertheless, it is possible to obtain an analytical approximation that provides satisfactory estimates for the constraints and that can be used as *initial values* for further refinements of an iterative search. We illustrate this with the following example.

Consider the system, Eqs. (1a) and (1b), prepared in the state **A** $[x(0) = x_0, y(0) = y_0]$ subject to a variation of the control parameter such that the asymptotic fixed point be **B** ($x_{ss,\mathbf{B}} = P_\infty - 1$, $y_{ss,\mathbf{B}} = 1$), with the following functional form for the control parameter:

$$P(t) = P_b\theta(-t) + [a_0 + a_1 t + a_2 t^2]\theta(t)\theta(t_f - t) + P_\infty\theta(t - t_f)\,, \tag{4}$$

where P_b is the control parameter value for the initial condition, t_f, which now takes the role of $\bar{t}$, is the time at which x and y must reach *simultaneously* their asymptotic value, θ is the Heaviside function, and a_0, a_1, and a_2 are coefficients (to be determined) which provide a quadratic form for the control parameter. In order to proceed to an estimate of the a_j's we make the following assumptions, very well verified in practice (for all those systems with $\epsilon \ll 1$): x takes negligibly small values over most of the time interval; x grows very rapidly towards its final value; we can calculate the evolution of y in the linear approximation. The error that we make by adopting this approximation for an estimate of $y(t)$ is small since x takes non-negligible values only over a very short time (typically less than 1% of the evolution) [11], and we aim for a direct transition towards **B** without overshoot and spiraling (i.e., x does not become so large that the approximation may be broken even in a very short time).

Under these approximations Eq. (1b) reduces to

$$\dot{y} = -\epsilon(y - P)\,, \tag{5}$$

which can be integrated to provide

$$y(t) = \left[y_0 - a_0 + \frac{a_1}{\epsilon} - 2\frac{a_2}{\epsilon^2}\right]e^{-\epsilon t} + a_0 + a_1\left(t - \frac{1}{\epsilon}\right) + a_2\left(t^2 - 2\frac{t}{\epsilon} + \frac{2}{\epsilon^2}\right), \tag{6}$$

where we have made use of the initial condition $y(t = 0) = y_0$. Using Eq. (6) we can explicitly calculate the rhs of Eq. (3) and use the equality with the lhs to determine the coefficients of $P(t)$ [Eq. (4)] which satisfy the integral [Eq. (3)] at the shortest possible time t_f.

From conditions $P(t = 0^+) = P_0$ and $P(t_f) = P_f$ (both arbitrarily fixed) we obtain immediately $a_0 = P_0$, $a_1 = (P_\infty - y_0 + a_2 t_f^2)/t_f$, and

$$a_2 = \frac{\log(\frac{x_{ss,\mathbf{B}}}{x_0}) - (P_0 - 1)t_f + (P_0 - P_f)(\frac{t_f}{2} - \frac{1}{\epsilon}) + \frac{1}{\epsilon}(P_0 - y_0 + \frac{P_0 - P_f}{\epsilon t_f})(1 - e^{-\epsilon t})}{(\frac{t_f}{\epsilon^2} - \frac{2}{\epsilon^3})(1 - e^{-\epsilon t}) - \frac{1}{6}t_f^3 - \frac{t_f^2}{\epsilon} + \frac{2}{\epsilon^2}t_f}\,. \tag{7}$$

If we fix $P_0 = 1.48$, $P_f = 0.85$, $t_f = 1.85 \times 10^{-4}$ s, $P_\infty = 1.1$ and $y_0 = 0.9 (= P_b)$, we obtain the following coefficients: $a_0 = 1.48$, $a_1 = -5004.58$, and $a_2 = 8.64421 \times 10^6$. By using these values as initial estimates, we optimize numerically and find the excellent result shown in Fig. 4. We remark that the amplitude of the residual oscillation (not visible in the figure) is approximately $10^{-4} \times x_{ss,\mathbf{B}}$ and that it represents an improvement of about 6 orders of magnitude compared to the unsteered transition (cf. Fig. 3; notice also the considerable reduction in the transition time). The error with which we have analytically estimated a_1 and a_2 is 4% and 6%, respectively, compared to the optimal values of Fig. 4. We also remark that this model is particularly stiff and that real systems display a much smaller number of relaxation oscillations (cf., e.g., [6]). Hence, the degree of precision with which the coefficients of the control parameter have to be known is correspondingly reduced, thereby increasing the applicability of the technique.

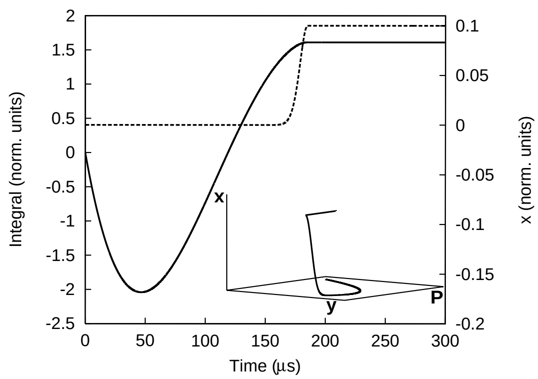

FIG. 4. Optimized steering of the transition. Integral (solid line) and x (dashed line) as a function of time for $a_0 = 1.5$, $a_1 = -5228.59$, and $a_2 = 9.21135 \times 10^6$. Notice that these optimized values require a slightly different value of $t_f = 1.85175 \times 10^{-4}$ s. At $t = t_f$ the two sides of Eq. (3) are equal, thus no oscillations ensue. The inset shows, in the 3D space, the direct evolution into the fixed point. Cf. Fig. 3 also comparing the horizontal and vertical scales.

Finally, we stress the fact that for the sake of an easy demonstration we have chosen a simple function for $P(t)$, but the choice is far from being unique. Indeed, the shape of $P(t)$, chosen compatibly with Eq. (3), can contain *a priori* a number of parameters as large as one wishes. To determine them, it will suffice to find additional constraints (e.g., specific points through which the trajectory has to pass during the transition) to be imposed onto the system. We remark the great potential for applications of this technique, e.g., in the realm of telecommunications, where notable improvements in the quality of the response of very low-cost semiconductor lasers may allow their use as transmitters in low-cost devices (e.g., telephones, modems, etc.).

In summary, we have shown that the transition between states of a 2D separable nonlinear system is fully described by an invariant characteristic integral. The time evolution of this integral, which depends on the control parameter, determines the way the transition occurs. Information obtained from this quantity can be used to determine a functional dependence of the control parameter on time to achieve a desired form for the transition.

We are grateful to S. Balle, L. M. Hoffer, A. Politi, M. San Miguel, and J. R. Tredicce for discussions and encouragement. One of us (G. L. L.) is particularly indebted to G. P. Puccioni for computer assistance.

*Corresponding author.
Email address: lippi@inln.cnrs.fr
†Permanent address: Central R & D, Crolles site/Reliability, 850 Rue J. Monnet, F-38926 Crolles CEDEX, France.
Email address: frederic.monsieur@st.com

[1] See, e.g., the special issue on *Control of Chaos: New Perspective in Experimental and Theoretical Nonlinear Science*, edited by F. T. Arecchi *et al.* [Int. J. Bifurcation Chaos Appl. Sci. Eng. **8** (1998)], and references therein.

[2] See, e.g., D. Kaplan and L. Glass, *Understanding Nonlinear Dynamics* (Springer, Berlin, 1995).

[3] J. M. T. Thompson and H. B. Stewart, *Nonlinear Dynamics and Chaos* (Wiley, New York, 1986).

[4] J. R. Tredicce, F. T. Arecchi, G. L. Lippi, and G. P. Puccioni, J. Opt. Soc. Am. B **2**, 173 (1985).

[5] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986).

[6] P. A. Porta, L. M. Hoffer, H. Grassi, and G. L. Lippi, Int. J. Bifurcation Chaos Appl. Sci. Eng. **8**, 1811 (1998).

[7] Throughout this paper we consider, for numerical and analytical purposes, a finite but extremely small value of x_0 (e.g., $x_0 = 10^{-15}$) as an initial condition, such that both analytical and numerical integrations be meaningful. This value "simulates" fluctuations which perturb the system away from **A**. However, we do not intend to reproduce by these means any stochastic effect, of no concern for this work.

[8] It is trivial to generalize the invariant integral to dynamical systems of the form $\dot{x} = xf(p_j, y_k)$—where p_j are parameters (in arbitrary number), y_k are dynamical variables (in arbitrary number) satisfying other ordinary differential equations, and f is a function—which possess the same invariant integral, Eq. (2), with integrand $f(p_j, y_k)$.

[9] We do not examine the case of the node, since it is physically uninteresting.

[10] In all practical cases, it will suffice to reduce the amplitude of the oscillations below the noise level to make the transition appear to have occurred without oscillations.

[11] G. L. Lippi, P. A. Porta, L. M. Hoffer, and H. Grassi, Phys. Rev. E **59**, R32 (1999).