Noncommutative D-Brane in a Nonconstant NS-NS B Field Background

Pei-Ming Ho* and Yu-Ting Yeh†

Department of Physics, National Taiwan University, Taipei 106, Taiwan, Republic of China

(Received 26 June 2000)

We show that, when the field strength H of the NS-NS B field does not vanish, the coordinates x and momenta p of open string end points satisfy a set of mixed commutation relations among themselves. Identifying x and p with the coordinates and derivatives of the D-brane world volume, we find a new type of noncommutative space which is very different from those associated with a constant B field background.

PACS numbers: 11.25.Mj, 11.25.Sq

(1) Introduction.—In search of the final theory, the quantum theory of gravity has been the most outstanding problem. In particular, one expects a frenetic, fuzzy structure of spacetime at the Planck scale, where the classical notion of geometry breaks down. As a realization of this fuzziness of spacetime, noncommutative geometry has been conjectured to underlie many mysterious features of quantum gravity, and thus provide a physical regularization for ordinary quantum field theories. This hope has motivated studies of quantum spacetime since 1946 by Snyder [1] and Yang [2].

In the past few years there has been a growth in interest in noncommutative geometry, which appears in string theory in several different ways. To our knowledge the first paper on this topic is [3]. For an earlier focus on the use of noncommutative geometry in matrix theory compactifications, see, for instance, Ref. [4]. In this paper we follow [5,6] and find new types of noncommutative spaces which appear naturally in string theory as a description of the D-brane world volume.

In [7,8], it was proposed that the matrix theory compactified on a torus with constant three-form C field background should be described by a field theory living on a noncommutative space whose coordinates satisfy a noncommutative algebra of the form

$$[x^i, x^j] = i\theta^{ij}, \tag{1}$$

where $\theta^{ij} = RC^{-ij}$ and R is the light cone radius of X^-. As evidence, the Bogomolnyi-Prasad-Sommerfield spectrum on the quantum torus was given in [9,10], and this conjecture was later derived [11] from the discrete light cone quantization of the membrane action. Via string dualities, it follows that, in the background of a constant Neveu-Schwarz–Neveu-Schwarz (NS-NS) B field, the low energy field theory of a flat D-brane in flat spacetime lives on a noncommutative space described by (1), where

$$\theta^{ij} = -2\pi\alpha'(G^{-1}BM^{-1})^{ij}, \tag{2}$$

$$M_{ij} = G_{ij} - B_{ik}G^{kl}B_{lj}, \tag{3}$$

where G_{ij} is the spacetime metric viewed by closed strings. Here we assumed that the U(1) field strength $F = dA$ vanishes. In general, since $\mathcal{F} = B - F$ is the gauge invariant quantity, it is natural to replace B by $\mathcal{F}$ in (2). (For the relation between different noncommutativity due to different choices of background values, see [13].) The simplest way to derive this result is to quantize an open string ending on the D-brane [5,6,12]. This serves as direct evidence for the noncommutativity of D-brane world volume in the B field background. Later it was shown [14,15] that, for the sake of deriving end point commutation relations, it is sufficient to approximate the open string by a straight line stretched between its end points. This is equivalent to saying that we quantize the open string in the low energy limit ($\alpha' \to 0$). Other approaches for calculating the D-brane world volume noncommutativity can be found in, e.g., [16–18].

In this paper we consider the more general case of a curved D-brane in a curved spacetime with a nonconstant B field. Obviously, Eq. (2) will not continue to hold, because Eq. (1) may no longer satisfy the Jacobi identities. We will show that in the generic case θ will be replaced by a function depending not only on the coordinates x but also on the derivatives ∂. The D-brane world volume thus belongs to a new type of noncommutative spaces which is described by a mixed algebra of x and ∂.

(2) Generic case.—The bosonic part of the action for an open string ending on a D-brane in the background of a NS-NS B field is

$$S_B = \int d\tau\, L = \frac{1}{4\pi\alpha'}\int d^2\sigma\,[\eta^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + \epsilon^{\alpha\beta}\mathcal{F}_{ij}\partial_\alpha X^i\partial_\beta X^j], \tag{4}$$

where $\eta^{\alpha\beta} = \mathrm{diag}(1,-1)$, and $\epsilon^{01} = -\epsilon^{10} = 1$. (We have absorbed the dilaton factor in $G_{\mu\nu}$ and $\mathcal{F}_{ij}$.) We use X^i and X^a to denote longitudinal and transverse directions for the D-brane, respectively, and use X^μ for all spacetime directions. For simplicity we assume that $\mathcal{F}_{a\mu} = 0$, $G_{ia} = 0$, and $\mathcal{F}_{ij}$ is invertible.

The conjugate momentum of X^μ is

$$P_\mu = \frac{1}{2\pi\alpha'}[G_{\mu\nu}\dot{X}^\nu + \mathcal{F}_{\mu i}X'^i], \tag{5}$$

and the boundary conditions are

$$G_{ij}X'^j + \mathcal{F}_{ij}\dot{X}^j = 0, \qquad \dot{X}^a = 0. \tag{6}$$

In the limit $\alpha' \to 0$, the oscillation modes can be ignored since their energies are proportional to $1/\alpha'$. The bulk of the string is now determined by its boundary. In principle, one can try to solve the wave equations for X^μ and pick out the lowest energy mode which survives the limit $\alpha' \to 0$. Here we avoid the complexity by focusing on the low energy limit in which both $\dot{X}$ and X' are very small. This means that the string is very short and moves very slowly, so that the spacetime appears to be almost flat and $\mathcal{F}$ is almost constant. Therefore we can use the results in [5] for a flat background and see that Eq. (6) holds for all σ for the lowest energy mode. It follows that $P_a = 0$ and

$$P_i = \frac{1}{2\pi\alpha'}(\mathcal{F}_{il} - G_{ij}\mathcal{F}^{jk}G_{kl})X'^l, \tag{7}$$

where $\mathcal{F}^{ij}$ is the inverse matrix of $\mathcal{F}_{ij}$.

Since P_a vanishes, X^a will be just constant for the whole string, and so we will ignore them from now on. Let

$$\hat{\mathcal{F}} = \frac{1}{2\pi\alpha'}(\mathcal{F} - G\mathcal{F}^{-1}G); \tag{8}$$

then a shorthand of (7) is

$$P = \hat{\mathcal{F}}X'. \tag{9}$$

The symplectic two-form which determines the commutation relations among X and P is

$$\Omega = \int d\sigma\,(\mathbf{d}X^i\mathbf{d}P_i). \tag{10}$$

By using (9), and the identity $dF = 0$, we find

$$\Omega = \frac{1}{2}[\mathbf{d}X^T\hat{\mathcal{F}}\mathbf{d}X]_{\sigma=0}^{\sigma=\pi} - \frac{1}{2}\int d\sigma\,\hat{H}_{ijk}X'^i\mathbf{d}X^j\mathbf{d}X^k, \tag{11}$$

where $\hat{H}_{ijk} = \partial_i\hat{\mathcal{F}}_{jk} + \partial_j\hat{\mathcal{F}}_{ki} + \partial_k\hat{\mathcal{F}}_{ij}$. Note that in the large B limit $\hat{H}$ is just $(1/2\pi\alpha')$ times $H = dB$ induced on the D-brane.

(3) $\hat{H} = 0$ and fuzzy sphere.—While $\hat{H}$ may be nontrivial in spacetime, as long as its projection onto the D-brane vanishes, the second term in (11) vanishes, and the Poisson bracket $(\cdot,\cdot)$ for the end points of the open string at $\sigma = 0$ is

$$(X^i, X^j) = 2\pi\alpha' i\theta^{ij}, \tag{12}$$

where $\theta = \hat{\mathcal{F}}^{-1}$. The relation for the other end point differs only by a sign.

To quantize this system we need to replace the Poisson brackets $(\cdot,\cdot)$ by commutators $[\cdot,\cdot]$, but it requires some operator ordering such that the Jacobi identity is satisfied. We will only be concerned with the Poisson bracket in this paper.

An example is provided by the spherical D2-brane in S^3, where the metric of S^3 is

$$ds^2 = k\alpha'[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)], \tag{13}$$

and the field strength for the two-form NS-NS B field is

$$H \equiv dB = 2k\alpha'\sin^2\psi\sin\theta\, d\psi\, d\theta\, d\phi, \tag{14}$$

where k is an integer related to the radius of S^3 by $R = \sqrt{k\alpha'}$. For this H, we can choose B to be proportional to the volume form of the two-sphere parametrized by (θ, ϕ) on which the D2-brane wraps:

$$B = k\alpha'\left(\psi - \frac{\sin 2\psi}{2}\right)\sin\theta\, d\theta\, d\phi. \tag{15}$$

The one-form field strength on the D2-brane should be [19,20]

$$F = dA = \pi\alpha' n\sin\theta\, d\theta\, d\phi. \tag{16}$$

The energy of the D2-brane is locally minimized at

$$\psi = \frac{\pi n}{k} \tag{17}$$

for arbitrary integer $0 < n < k$ [19–21]. At those places,

$$\mathcal{F} = B - F = -k\alpha'\left(\frac{\sin 2\psi}{2}\right)\sin\theta\, d\theta\, d\phi. \tag{18}$$

The resulting Poisson bracket is thus

$$(\cos\theta, \phi) = -\frac{2\pi}{k}\frac{\cos\psi}{\sin\psi}, \tag{19}$$

which implies that the Cartesian coordinates satisfy the algebra of the fuzzy sphere [22],

$$(x_i, x_j) = \frac{2\pi}{k}\frac{\cos\psi}{\sin\psi}\epsilon_{ijk}x_k, \tag{20}$$

where

$$x_1 = \sin\theta\cos\phi, \qquad x_2 = \sin\theta\sin\phi, \qquad x_3 = \cos\theta. \tag{21}$$

In the large k limit, $\psi \ll 1$, it is $(x_i, x_j) \simeq \frac{2}{n}\epsilon_{ijk}x_k$. This is in agreement with [18,23]. For discussions on noncommutative gauge theories on fuzzy spheres, see, e.g., [24–27].

The reason why this approximation works is that from the flat space results we see that the length of the open string is related to its momentum. In the low energy limit, the momentum is very small and so the open string is very short, and it sees only a very small portion of the sphere, which looks almost flat. This also explains why the result of the commutation relation should be formally the same as the flat case. The first main result of this paper is that the same expressions for noncommutativity [Eqs. (1) and (2)] continue to work as long as $\hat{H}_{ijk}$ vanishes. For the formulation of a noncommutative gauge theory on a generic Poisson manifold, see Refs. [28,29].

(4) $\hat{H} \neq 0$ and new type of noncommutative spaces.—What happens if $\hat{H}_{ijk}$ is not zero? An approximate result for small $\mathcal{F}$ and slow variations of $\mathcal{F}$ and g were obtained in [30]. There the Jacobi identity for the algebra of X and P was checked to hold within the validity of this approximation. In the following we will give a very similar derivation, but arriving at a consistent algebra which is

valid in the low energy limit of open strings. Our task is to find the Poisson brackets among X and P at $\sigma = 0$ for the case $\hat{H} =$ const, such that the Poisson brackets satisfy Jacobi identity to all orders, and reduces to the previous result (1) when $\hat{H} = 0$.

We will simplify the derivation by assuming that X is linearly dependent on σ. This statement is not well defined with respect to general coordinate transformations, so the results we obtain are exactly correct only up to the first order in X' or P, such as in a low energy approximation.

By assumption, X' is independent of σ and

$$X(\sigma) = x + \sigma X', \tag{22}$$

where x is the coordinate of the end point of the string at $\sigma = 0$. In our convention, $\sigma \in [0, \pi]$. The momentum at $\sigma = 0$ is

$$p = P(\sigma = 0) = \hat{\mathcal{F}}(x) X'. \tag{23}$$

From (11), assuming that $\partial_k \hat{\mathcal{F}}_{ij} =$ const so that $\hat{H}_{ijk} =$ const, the symplectic two-form is

$$\begin{aligned}\Omega = {}& \frac{\pi}{2}(\partial_k \hat{\mathcal{F}}_{ij} - \hat{H}_{ijk}) X'^k \mathbf{d}x^i \mathbf{d}x^j + \frac{\pi}{2}\left[\hat{\mathcal{F}}_{ij} + \pi\left(\partial_k \hat{\mathcal{F}}_{ij} - \frac{1}{2}\hat{H}_{ijk}\right) X'^k\right](\mathbf{d}x^i \mathbf{d}X'^j - \mathbf{d}x^j \mathbf{d}X'^i) \\ &+ \frac{\pi^2}{2}\left[\hat{\mathcal{F}}_{ij} + \pi\left(\partial_k \hat{\mathcal{F}}_{ij} - \frac{1}{3}\hat{H}_{ijk}\right) X'^k\right]\mathbf{d}X'^i \mathbf{d}X'^j,\end{aligned} \tag{24}$$

where $\hat{\mathcal{F}}_{ij} = \hat{\mathcal{F}}_{ij}(x)$. It can be explicitly checked that the symplectic two-form is closed, so that its inverse, the Poisson bracket, satisfies the Jacobi identity. By inverting Ω, we obtain the Poisson brackets for (x, x), (x, X'), and (X', X'). To find (x, p) and (p, p) from these, we use (23).

Since it is straightforward but cumbersome to write the final answer to all of the Poisson brackets among x and p, we will write only the one involving x for the special case $\partial_i \hat{\mathcal{F}}_{jk} = \hat{H}_{ijk}/3$, that is, $\hat{\mathcal{F}}_{ij}(x) = \hat{\mathcal{F}}^{(0)}_{ij} + \frac{1}{3}\hat{H}_{ijk} x^k$, where $\hat{\mathcal{F}}^{(0)}_{ij}$ are constant. The result is

$$(x^i, x^j) = -2([I + A]^{-2} \hat{\mathcal{F}}^{-1})^{ij}, \tag{25}$$

where I stands for the identity matrix and

$$A^i_j = \frac{\pi}{6} \hat{\mathcal{F}}^{-1ik} \hat{H}_{kjm} \hat{\mathcal{F}}^{-1mn} p_n . \tag{26}$$

When $\hat{\mathcal{F}}^{(0)}$ is much larger than $\hat{H}$, (25) is approximately

$$(x^i, x^j) = -2\left[\delta^i_l - \frac{\pi}{3} \hat{\mathcal{F}}^{-1ik} \hat{H}_{klm} \hat{\mathcal{F}}^{-1mn} p_n\right] \hat{\mathcal{F}}^{-1lj}. \tag{27}$$

The commutation relation for the coordinates at the other end point of the open string at $\sigma = \pi$ is the same, except for a difference in sign, as it should be [5]. These expressions show that, after quantization, the commutator of x with x will in general be a function of x and p.

It follows that the low energy D-brane field theory lives on a noncommutative space. Identifying x and p with the coordinates and derivatives on the D-brane, the commutation relations among x and p define the differential calculus on its noncommutative world volume. The novel property that comes in when $\hat{H} \neq 0$ is that the commutator $[x^i, x^j]$ is given by a function of x and p, that is, a pseudodifferential operator on the noncommutative space. Similarly, the commutator of $[x, p]$ and $[p, p]$ is also given by functions of x and p, rather than just a function of x. These types of noncommutative spaces were not considered in the context of string theory in the recent past, but were considered a long time ago [1,2]. [The motivation of Refs. [1,2] to consider noncommutative spaces was to regularize ordinary quantum field theories. In order to have Lorentz invariance, mixing of coordinates and momenta is needed. For instance, they have $[x^i, x^j] = ia^2(\{x^i, p^j\} - \{x^j, p^i\})$.] Recently, in [31–33], similar noncommutative spaces (fuzzy S^4) were considered in matrix model and M theory.

More care is needed to define a field theory on such noncommutative spaces. Since the commutator of two spacetime coordinates generates a derivative, how do we distinguish a function of x only from a function of both x and $\partial/\partial x$ on the noncommutative space? This problem can be solved by requiring that a function of x be written in terms of totally symmetrized products of the x's. However, it is not clear how to define a gauge theory, since the gauge transformation of a field, which is a function of x, will generically become a pseudodifferential operator. Perhaps the generalization of gauge theories to such noncommutative spaces demands a deeper understanding of gauge symmetry. On the other hand, despite this difficulty in defining a noncommutative gauge theory, we should not be surprised that these types of noncommutative spaces appear in string theory, since, in string theory, operators which are identified with coordinates or momenta may be reinterpreted as other physical quantities in a dual theory.

From (23) and (24), one can also find (x^i, p_j) as an expansion of momentum. Following the lines of [30], it may be possible to derive an uncertainty relation in the absence of background fields of the form $\Delta x \Delta p \geq \frac{1}{2} + \frac{a^2}{2}\Delta p \Delta p + \ldots$, due to quantum fluctuations. These types of uncertainty relations were known to result in a minimal length $\Delta x \geq a$, which cannot be obtained for ordinary noncommutative spaces. This possibility suggests that the mixing of x and p may play an essential role in the noncommutative structure of spacetime at the Planck scale.

The approach used in this paper should work even for cases in which $\hat{H}$ is not constant, although it will be more difficult to obtain generic expressions for the symplectic form unless more details about $\hat{H}$ are specified.

An open membrane ending on an M5-brane in the background of a three-form C field with constant field strength was studied in [34] in a limit in which the boundary of the open membrane—a closed string— gives a noncommutative loop algebra on the 5-brane world volume. This can also be interpreted as the noncommutativity felt by a fundamental closed string in the background of a constant H field. It would be interesting to see the connection between the noncommutativities from the open and closed string points of view.

We thank Chong-Sun Chu and Miao Li for valuable discussions. This work is supported in part by the National Science Council, Taiwan, Republic of China, and the Center for Theoretical Physics at National Taiwan University.

*Electronic address: pmho@phys.ntu.edu.tw
†Electronic address: r7222049@ms.cc.ntu.edu.tw

[1] H. S. Snyder, Phys. Rev. **71**, 38 (1946); Phys. Rev. **72**, 68 (1947).
[2] C. N. Yang, Phys. Rev. **72**, 874 (1947).
[3] E. Witten, Nucl. Phys. **B268**, 253 (1986).
[4] P.-M. Ho and Y.-S. Wu, Phys. Lett. B **398**, 52 (1997); P.-M. Ho, Y.-Y. Wu, and Y.-S. Wu, Phys. Rev. D **58**, 026006 (1998); P.-M. Ho and Y.-S. Wu, Phys. Rev. D **58**, 066003 (1998); P.-M. Ho and Y.-S. Wu, Phys. Rev. D **60**, 026002 (1999).
[5] C.-S. Chu and P.-M. Ho, Nucl. Phys. **B550**, 151 (1999).
[6] C.-S. Chu and P.-M. Ho, hep-th/9906192.
[7] A. Connes, M. R. Douglas, and A. Schwarz, J. High Energy Phys. **02**, 003 (1998).
[8] M. R. Douglas and C. Hull, J. High Energy Phys. **02**, 008 (1998).
[9] P.-M. Ho, Phys. Lett. B **434**, 41 (1998).
[10] D. Brace, B. Morariu, and B. Zumino, Nucl. Phys. **B545**, 192 (1999).
[11] C.-S. Chu, P.-M. Ho, and M. Li, hep-th/9911153.
[12] F. Ardalan, H. Arfaei, and M. M. Sheikh-Jabbari, hep-th/9803067; hep-th/9810072.
[13] N. Seiberg and E. Witten, J. High Energy Phys. **9909**, 032 (1999).
[14] D. Bigatti and L. Susskind, hep-th/9908056.
[15] Z. Yin, Phys. Lett. B **466**, 234 (1998).
[16] V. Schomerus, J. High Energy Phys. **9906**, 030 (1999).
[17] H. Garcia-Compean and J. F. Plebanski, hep-th/9907183.
[18] A. Yu. Alekseev, A. Recknagel, and V. Schomerus, hep-th/9908040; hep-th/0003187.
[19] C. Bachas, M. Douglas, and C. Schweigert, hep-th/0003037.
[20] W. Taylor, hep-th/0004141.
[21] A. Kling, M. Kreuzer, and J.-G. Zhou, hep-th/0005148.
[22] J. Madore, *An Introduction to Noncommutative Differential Geometry and Its Physical Applications* (Cambridge University Press, Cambridge, UK, 1999), 2nd ed.
[23] J. Pawelczyk, hep-th/0003057.
[24] J. Madore, S. Schraml, P. Schupp, and J. Wess, hep-th/0001203.
[25] E. Hawkins, q-alg/9708030.
[26] C. Klimčik, hep-th/9710153.
[27] U. Carow-Watamura and S. Watamura, hep-th/9801195.
[28] T. Asakawa and I. Kishimoto, hep-th/0002138.
[29] B. Jurčo, P. Schupp, and J. Wess, hep-th/0005005.
[30] C.-S. Chu, P.-M. Ho, and Y.-C. Kao, hep-th/9904133.
[31] J. Castelino, S. Lee, and W. Taylor, Nucl. Phys. **B526**, 334 (1998).
[32] M. Berkooz and H. Verlinde, J. High Energy Phys. **9911**, 037 (1999).
[33] P.-M. Ho and M. Li, hep-th/0004072.
[34] E. Bergshoeff, D. S. Berman, J. P. van der Schaar, and P. Sundell, hep-th/0005026.