Is Equilibrium of Aligned Kerr Black Holes Possible?

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(Received 14 September 2000)

We show that equilibrium of two Kerr black holes can be achieved by placing between them a relativistic disk or a third Kerr black hole, the latter case demonstrating the existence of equilibrium configurations in the purely black hole systems with the number of constituents more than two.

PACS numbers: 04.70.Bw, 04.20.Jb, 97.60.Lf

A thorough analysis of the famous double-Kerr solution of Kramer and Neugebauer [1] leads to the conclusion, rigorously proved in the equatorially symmetric case and extensively supported by a numerical study in the general case [2], that two Kerr black holes cannot be in equilibrium due to the balance of the gravitational attraction and spin-spin repulsion forces.

A natural question arises: Is it possible to achieve equilibrium of two stationary black holes by introducing a third component? Although no attempt has yet been done to solve any concrete equilibrium three-body problem involving normal black holes, apparently because of the complexity of the corresponding balance equations, the mathematically equivalent exact solutions of Einstein's equations are known [3] which can describe an axisymmetric system of N aligned Kerr black holes. In this Letter we use the analytically extended version of the $2N$-soliton solution [4] whose very concise analytic form will enable us to demonstrate that two normal Kerr black hole constituents can be equilibrated by placing between them a superextreme object (relativistic disk) or, most unexpectedly, a third black hole constituent, the latter case representing a purely gravitational balance of three stationary black holes with positive individual masses.

For our specific three-body problem we make use of a six-soliton specialization of the solution [4] whose defining Ernst complex potential $\mathcal{E}$ [5] and corresponding metric functions f, γ, and ω from the axisymmetric line element

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega\, d\varphi)^2, \tag{1}$$

ρ, z, φ, t being the Weyl-Papapetrou cylindrical coordinates and time, have the form (a bar over a symbol denotes complex conjugation; throughout the paper units are used in which the speed of light c and the gravitational constant G are equal to unity)

$$\begin{aligned}
&\mathcal{E} = \frac{E_+}{E_-}, \qquad f = \frac{E_+\bar{E}_- + \bar{E}_+E_-}{2E_-\bar{E}_-}, \qquad e^{2\gamma} = \frac{E_+\bar{E}_- + \bar{E}_+E_-}{2\lambda\bar{\lambda}\prod_{n=1}^{6} r_n}, \qquad \omega = \frac{2i(G\bar{E}_- - \bar{G}E_-)}{E_+\bar{E}_- + \bar{E}_+E_-},\\
&E_\pm = \Lambda \pm \Gamma, \qquad \Lambda = \sum_{1\le i<j<k\le 6} \lambda_{ijk} r_i r_j r_k, \qquad \Gamma = \sum_{1\le i<j\le 6} \nu_{ij} r_i r_j,\\
&G = -\sigma\Lambda + \bar{\sigma}\Gamma + z\Gamma + \sum_{1\le i<j<k\le 6} \lambda_{ijk}(\alpha_i + \alpha_j + \alpha_k) r_i r_j r_k\\
&\qquad - \sum_{1\le i<j\le 6} \nu_{ij}(\alpha_{i'} + \alpha_{j'} + \alpha_{k'} + \alpha_{l'}) r_i r_j, \qquad (i',j',k',l' \neq i,j; i'<j'<k'<l'),\\
&\lambda_{ijk} := (-1)^{i+j+k} V(\alpha_i,\alpha_j,\alpha_k) V(\alpha_{i'},\alpha_{j'},\alpha_{k'}) X_i X_j X_k, \qquad (i',j',k' \neq i,j,k; i'<j'<k'),\\
&\nu_{ij} := (-1)^{i+j}(\alpha_i - \alpha_j) V(\alpha_{i'},\alpha_{j'},\alpha_{k'},\alpha_{l'}) X_i X_j, \qquad (i',j',k',l' \neq i,j; i'<j'<k'<l'),\\
&V(\alpha_1,\ldots,\alpha_k) := \prod_{1\le i<j\le k}(\alpha_i - \alpha_j), \qquad X_i := \frac{(\alpha_i - \bar{\beta}_1)(\alpha_i - \bar{\beta}_2)(\alpha_i - \bar{\beta}_3)}{(\alpha_i - \beta_1)(\alpha_i - \beta_2)(\alpha_i - \beta_3)}, \qquad r_i := \sqrt{\rho^2 + (z-\alpha_i)^2},\\
&\sigma := \sum_{l=1}^{3}\beta_l = \Big[\nu + \sum_{1\le i<j<k\le 6}\lambda_{ijk}(\alpha_i+\alpha_j+\alpha_k)\Big]\Big/\lambda, \qquad \nu := \sum_{1\le i<j\le 6}\nu_{ij}, \qquad \lambda := \sum_{1\le i<j<k\le 6}\lambda_{ijk}.
\end{aligned} \tag{2}$$

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Formulas (2) involve the parameters β_l, $l = 1, 2, 3$, which can assume arbitrary complex values, and the parameters α_n, $n = 1, \ldots, 6$, which can assume arbitrary real values or occur in complex conjugate pairs; they permit a simultaneous treatment of the subextreme (black hole) and superextreme (relativistic disk) cases.

For simplicity we consider the equatorially symmetric three-body problem defined, as can be readily demonstrated using the results of Ref. [4], by the following restrictions on the parameters α_n and β_l:

$$\alpha_4 = -\alpha_3, \qquad \alpha_5 = -\alpha_2, \qquad \alpha_6 = -\alpha_1, \qquad X_1X_6 = X_2X_5 = X_3X_4 = -1. \tag{3}$$

The two cases of interest are shown in Fig. 1 where the segments $\alpha_2 < |z| < \alpha_1$ are Killing horizons of two identical Kerr black holes, and the cut joining the points α_3 and $-\alpha_3$ in Fig. 1A defines a superextreme Kerr constituent (a relativistic disk), while the segment $|z| < \alpha_3$ in Fig. 1B is the horizon of the third Kerr black hole.

By construction, the solution (2)–(3) is asymptotically flat which means that the metric functions γ and ω are zeros on the part $|z| > \alpha_1$ of the symmetry axis. Therefore, in view of the additional symmetry of our three-body systems with respect to the equatorial ($z = 0$) plane we should demand only

$$\gamma(\rho = 0, \mathrm{Re}\alpha_3 < z < \alpha_2) = 0, \qquad \omega(\rho = 0, \mathrm{Re}\alpha_3 < z < \alpha_2) = 0 \tag{4}$$

to assure the equilibrium of all the constituents due to the balance of the gravitational attraction and spin-spin repulsion forces [6,7]. Since in all the formulas (2) one can use the constant objects X_n, $n = 1, \ldots, 6$, instead of the parameters β_l, $l = 1, 2, 3$, the system (4) can be solved numerically by assigning particular values to α_1, α_2, α_3, X_1 and finding the respective values of X_2 and X_3 [X_4, X_5, X_6, as well as α_4, α_5, α_6 are defined by (3)]. The explicit form of the system (4) is this:

$$\begin{aligned} &A_{12}[A_{13}^2A_{23}^2(X_1^2X_2^2X_3^2 + 1) + B_{13}^2B_{23}^2(X_1^2X_2^2 + X_3^2)] + \\ &4\alpha_3B_{12}[\alpha_1A_{23}B_{23}X_1X_3(X_2^2 + 1) - \alpha_2A_{13}B_{13}X_2X_3(X_1^2 + 1)] = 0, \\ &B_{12}\{\alpha_1A_{13}B_{13}X_1[B_{23}^2(X_2^2 - X_3^2) + A_{23}^2(X_2^2X_3^2 - 1)] - \alpha_2A_{23}B_{23}X_2[B_{13}^2(X_1^2 - X_3^2) + \\ &A_{13}^2(X_1^2X_3^2 - 1)] + 4\alpha_1\alpha_2\alpha_3X_3[A_{23}B_{23}X_1(X_2^2 - 1) - A_{13}B_{13}X_2(X_1^2 - 1)]\} + \\ &A_{12}[A_{13}^2A_{23}^2(B_{12} + \alpha_3)(X_1^2X_2^2X_3^2 - 1) + B_{13}^2B_{23}^2(B_{12} - \alpha_3)(X_1^2X_2^2 - X_3^2)] = 0, \\ &A_{ij} := \alpha_i - \alpha_j, \qquad B_{ij} := \alpha_i + \alpha_j, \end{aligned} \tag{5}$$

and below we give two particular solutions of the balance equations together with the corresponding individual masses M_i and angular momenta J_i, $i = 1, 2, 3$, of the constituents (the subindex 2 denotes the intermediate object) which have been calculated via Komar integrals [8] using Tomimatsu's formulas [9].

(A) An equilibrium of two black holes and a relativistic disk.—

$$\begin{aligned} &\alpha_1 = 4, \quad \alpha_2 = 3.5, \quad \alpha_3 = -6i, \quad X_1 = 0.6 + 0.8i, \\ &X_2 = 0.849 - 0.528i, \quad X_3 = 0.109i, \\ &M_1 = M_3 = 0.348, \quad M_2 = 1.176, \quad J_1 = J_3 = -0.116, \quad J_2 = 8.267 \end{aligned} \tag{6}$$

(the numerical values are given up to three decimal places).

(B) An equilibrium of three Kerr black holes.—

$$\begin{aligned} &\alpha_1 = 5, \quad \alpha_2 = 2, \quad \alpha_3 = 1, \quad X_1 = 0.92 - 0.392i, \\ &X_2 = 0.769 - 0.639i, \quad X_3 = 0.699 - 0.715i, \\ &M_1 = M_3 = 3.533, \quad M_2 = 1.813, \quad J_1 = J_3 = 70.66, \quad J_2 = -128.784. \end{aligned} \tag{7}$$

A peculiar feature of both equilibrium states is that the intermediate object is counterrotating to the two identical black hole constituents although there are examples when it is corotating. One could also observe that, whereas the black hole constituents in the equilibrium position (6) are "normal" black holes in the sense that they satisfy the inequality $J_{1,3}^2/M_{1,3}^4 < 1$ valid for a single Kerr black hole [10], it is tempting to view the black holes in the equilibrium position (7) as superextreme objects because of the inequality $J_i^2/M_i^4 > 1$ they fulfil. As a matter of fact, it is known [11] that already in a binary system of identical Kerr black holes the total angular momentum per unit mass can exceed considerably the total mass. Hence, in the multiblack hole solutions the ratio between the mass and

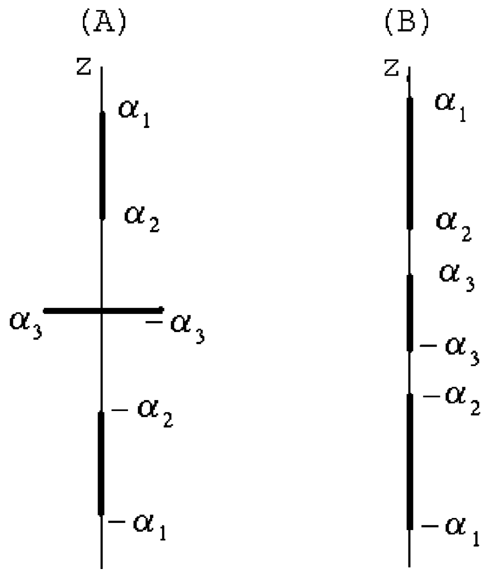

FIG. 1. Equatorially symmetric three-body systems composed (A) of two identical black holes and a relativistic disk and (B) of three black hole constituents, the upper and lower ones being identical.

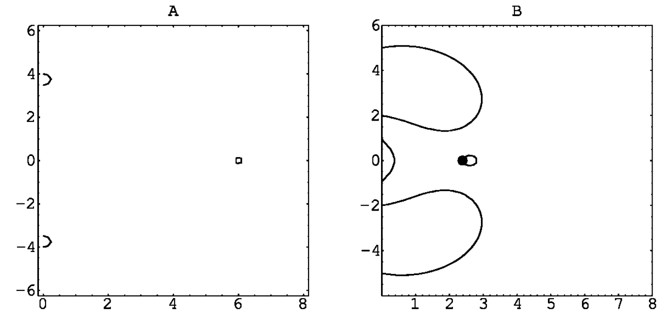

FIG. 2. Stationary limit surfaces plotted in the ρ (horizontal axis) and z (vertical axis) coordinates for (A) the equilibrium configuration (6) of two black holes and a relativistic disk, and (B) the equilibrium configuration (7) of three black holes (the point $z = 0$, $\rho = 2.388$ is a ring singularity).

angular momentum does not work as the absolute characteristic of the black hole state. What really matters is the presence of the horizons defined by real-valued parameters α_n (a pair of complex conjugate αs defines a superextreme object).

In Fig. 2 we have plotted the stationary limit surfaces for the above two equilibrium configurations. The case involving a superextreme object (Fig. 2A) does not have ring singularities, while the equilibrium configuration of three black holes possesses one ring singularity in the equatorial plane (Fig. 2B). The singularity is massless and its origin is most likely due to the rupture of the stationary limit surface of the middle black hole caused by a big value of the angular momentum. It is anticipated that the singularity has a "benign character," using Wald's wording [12], unlike the one discussed in the first paper of Ref. [2] where it accompanied a constituent with negative Komar mass. In any case, one might expect new physics inherent to the multiblack hole systems since no analogous equilibrium configuration of two Kerr black holes with positive Komar masses and a ring singularity is known.

Therefore, we can speak about a reliable mechanism of achieving equilibrium of two black holes by placing a third object between them. Remarkably, the latter can be a black hole constituent, giving rise to the multiblack hole equilibrium configurations. The fact that purely black hole equilibrium states do not appear in binary systems and do appear in three-body systems is probably explained by the existence of a minimal set of the parameters determining the interaction in a many-body system (these are sometimes associated with the individual Newman-Unti-Tamburino parameters of the constituents [7]) necessary to satisfy the balance equations and positive Komar mass conditions which is absent in the two-body case.

This work has been partially supported by Project No. 34222-E from CONACyT of Mexico and by Project No. PB96-1306 from DGICyT of Spain.

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