Monitor-Outside-a-Monitor Effect and Self-Similar Fractal Structure in the Eigenmodes of Unstable Optical Resonators

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A novel mechanism is proposed for the generation of self-similar structure over a limited range of length scales. Our mechanism, which we call the monitor-outside-a-monitor effect, comprises repeated magnification and addition of small-scale structure. We invoke this mechanism to explain recent observations of fractal structure in the eigenmodes of unstable optical resonators [G. P. Karman *et al.*, Nature (London) **402**, 138 (1999)].

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I. Introduction.—Recent computer simulations have shown that intensity cross sections through the lowest-loss eigenmodes of canonical unstable laser resonators are fractals [1–3]. Self-similar fractal structure was also shown to be present in noncanonical resonators specifically designed to have fractal eigenmodes [4].

That the eigenmodes of canonical resonators possess fractal structure was an exciting find. It was supported by both evidence from box counting and an outline for a possible explanation: A round-trip through an unstable canonical resonator magnifies (i.e., stretches) a light beam. But as eigenmodes are unchanged after one round-trip, the eigenmodes of unstable canonical resonators must be magnified copies of themselves, i.e., fractals.

The details of this mechanism were left unclear; instead it was pointed out that the detailed patterns arise from complex processes that have to be tackled with numerical techniques [1]. In this paper we describe a new mechanism for generating fractal structure and apply it to explain in detail how fractal structure arises in the eigenmodes of unstable resonators. Because of its parallels with video feedback we call this mechanism the *monitor-outside-a-monitor effect*.

II. Monitor-inside-a-monitor effect and monitor-outside-a-monitor effect.—Pointing a video camera at the monitor that displays the currently recorded image causes video feedback (for a good overview, see Ref. [5]). Figure 1 sketches the first four iterations of the *monitor-inside-a-monitor effect,* a well-known video-feedback phenomenon that occurs when the overall magnification of the camera-monitor combination, M, is chosen such that the monitor displays a demagnified image of itself, i.e., when $|M| < 1$.

In the case $|M| > 1$ nothing interesting appears to happen; the central section of the screen simply gets magnified until it fills the entire screen. However, if the resolution of the imaging system is high enough to resolve the grid of individual pixels, this periodic array of small structures is repeatedly imprinted on the pattern. From this small structure the magnification in the system successively forms structures $M, M^2, M^3, \ldots$ times larger than the original structure, while fresh small structure is repeatedly added. We call this effect the monitor-outside-a-monitor effect. Whereas the monitor-inside-a-monitor effect starts with large structures and adds smaller and smaller structures, the monitor-outside-a-monitor effect starts with small structures and adds successively larger structures.

In this paper we concentrate on a simplified description of the monitor-outside-a-monitor effect in one dimension, which iteratively stretches functions and adds to the result a periodic (or almost periodic) *pixel function*, $p(x)$. Starting with a function $f_0(x) \equiv 0$, the function $f_n(x)$ resulting from n iterations is the sum of the pixel function, the pixel function stretched by M, the pixel function stretched by M^2, and so on up to the pixel function stretched by M^{n-1}, so [6]

$$f_n(x) = \sum_{i=0}^{n-1} p\left(\frac{x}{M^i}\right). \tag{1}$$

This equation describes both the monitor-inside-a-monitor effect and the monitor-outside-a-monitor effect, the former for $|M| < 1$, the latter for $|M| > 1$. It describes functions that are very closely related to Weierstrass functions [7,8], the first example of functions that cannot be differentiated at any point.

Figure 2 shows the monitor-outside-a-monitor effect at work with a one-dimensional, periodic, pixel function, $p(x) = -\cos(\pi x/w)$. This simple function, which contains only structures of one size, w, is successively turned into a complex pattern, any part of which contains structures of sizes $w, Mw, M^2w, \ldots$. This property is a hallmark of fractals.

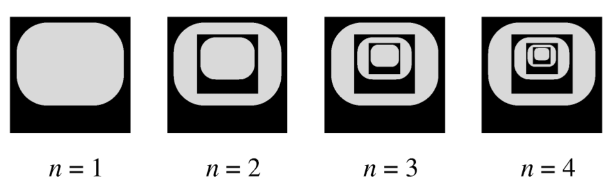

FIG. 1. Patterns generated by n iterations of the monitor-inside-a-monitor effect. In this example the magnification factor is $M = 1/2$.

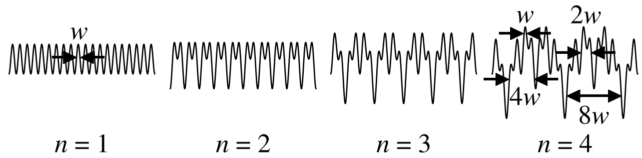

FIG. 2. Function $f_n(x)$ resulting from n iterations of the one-dimensional monitor-outside-a-monitor effect with a magnification $M = 2$ and a pixel function $p(x) = -\cos(\pi x/w)$. The sizes contained within $f_4(x)$ are indicated in multiples of the structure size w in the pixel function.

Clearly the patterns generated by the monitor-outside-a-monitor effect are not *mathematical,* i.e., perfect, fractals, as they contain no structures smaller than s. They do, however, contain self-similar structure over a finite range of length scales, just like all *physical* fractals, i.e., fractals that actually occur in the natural world [9,10]. The following section explores how the monitor-outside-a-monitor effect shapes the light in unstable canonical resonators into such physical fractals.

III. Monitor-outside-a-monitor-effect in unstable canonical resonators.—In unstable canonical resonators planes exist that are imaged onto themselves after one round-trip through the resonator. These *self-conjugate* planes can be real or virtual and can even be located outside the resonator. For each self-conjugate plane that is imaged onto itself with magnification M, another self-conjugate plane (corresponding to light traveling in the opposite direction) exists that is imaged onto itself with magnification $1/M$.

Throughout this paper one-dimensional confocal resonators of the type shown in Fig. 3 are used as convenient examples of canonical resonators. The self-conjugate focal plane with associated magnification M of modulus $|M| > 1$ will be referred to as S_M; the other self-conjugate focal plane, which has an associated magnification of $1/M$, is called $S_{1/M}$.

We note that the existence of these self-conjugate planes immediately explains two important properties of unstable resonators: First, the beam cross section in the plane S_M is magnified upon each round-trip through the resonator until the beam is so large that its edges are clipped by the aperture. It is this mechanism that makes unstable resonators lossy. Second, the beam cross section in the plane $S_{1/M}$ is demagnified upon each round-trip until it is essentially a diffraction-limited point. In a ray-optical picture, every light ray passes through this point. Wave-optically, this point is at the focus of the approximately spherical phase fronts that characterize the eigenmodes of unstable resonators [11].

Apertures in unstable resonators do not only lead to loss; they also modify the small-scale structure of the part of the beam that passes through the aperture. To develop a model for the combination of this effect of the aperture and the magnification of the beam in the plane S_M, we consider a uniform amplitude distribution in the plane S_M. After one round-trip through the resonator, the amplitude distribution can be described as the sum of an oscillating part, $p(x)$, and a dc offset [Fig. 4(a)]. During the next round-trip the oscillating part is simply stretched by a factor M [Fig. 4(b)], while the dc offset is turned into $p(x)$ and a new dc offset, so the amplitude distribution is now the sum of a dc offset, and a structure part, $s(x)$, which is the sum of $p(x)$ and $p(x/M)$. Just as in the case of $p(x)$ we choose to neglect the effect of the aperture on the structure part, an approximation that is justified by the accuracy of our model as described below. It follows that, starting with a uniform amplitude distribution in the plane S_M, the structure of the amplitude distribution after n round-trips can be

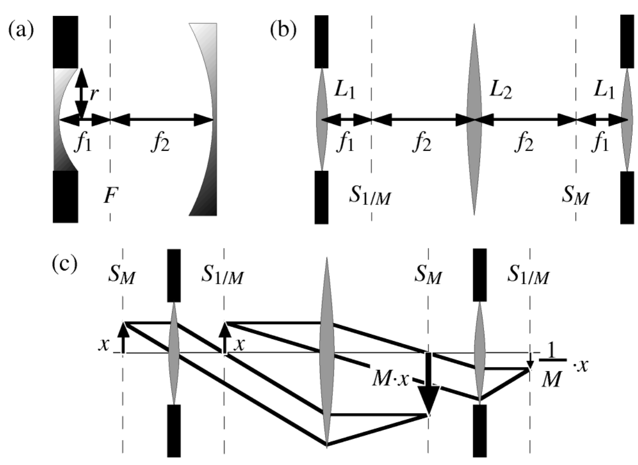

FIG. 3. Confocal resonator (a), its unfolded lens-guide equivalent (b), and imaging of the self-conjugate planes in the lens-guide equivalent (c). The left mirror (corresponding to lens L_1) has a hard-edged aperture of half-width r; the right mirror has no aperture. The plane of the common focus, F, unfolds into the two self-conjugate planes S_M and $S_{1/M}$. A second pair of self-conjugate planes is virtual and located at $\pm\infty$.

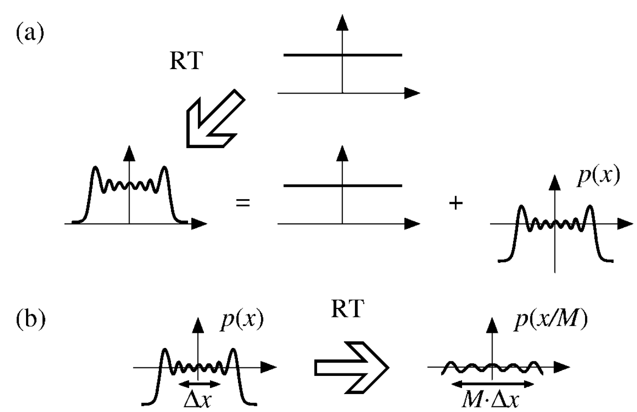

FIG. 4. Simplified description of the effect of round-trips (RTs) through the resonator on an initially uniform amplitude distribution in the plane S_M. (a) The diffraction pattern after one round-trip can be described as the sum of an oscillating part, $p(x)$, and another uniform amplitude distribution. (b) The oscillating part is simply stretched by a factor M.

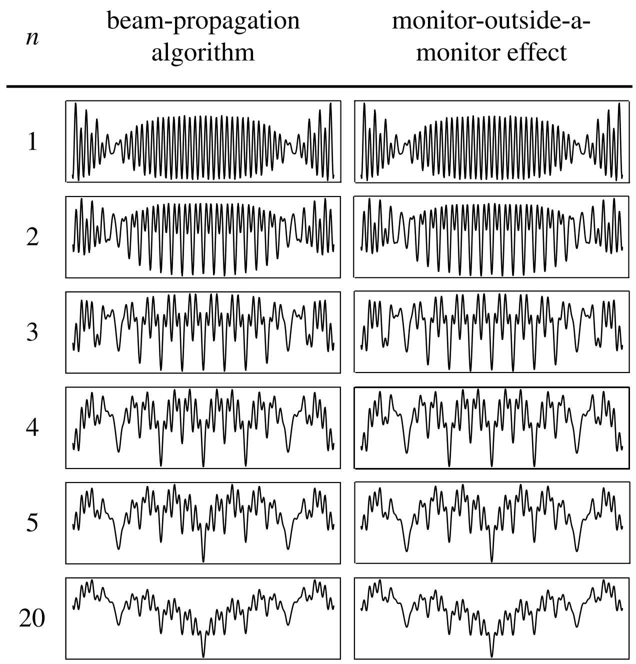

FIG. 5. Amplitude distributions over the central 0.47 mm in plane S_M in a beam passing trough a canonical resonator, simulated using a beam-propagation algorithm and the monitor-outside-a-monitor effect. The parameters of the resonator are $f_1 = 20$ mm, $f_2 = 40$ mm (leading to a magnification $M = -2$), and $r = 2.65$ mm, corresponding to a Fresnel number of $N \approx 92.5$. n is the number of round-trips; after $n = 20$ the beam cross section is essentially that of the eigenmode.

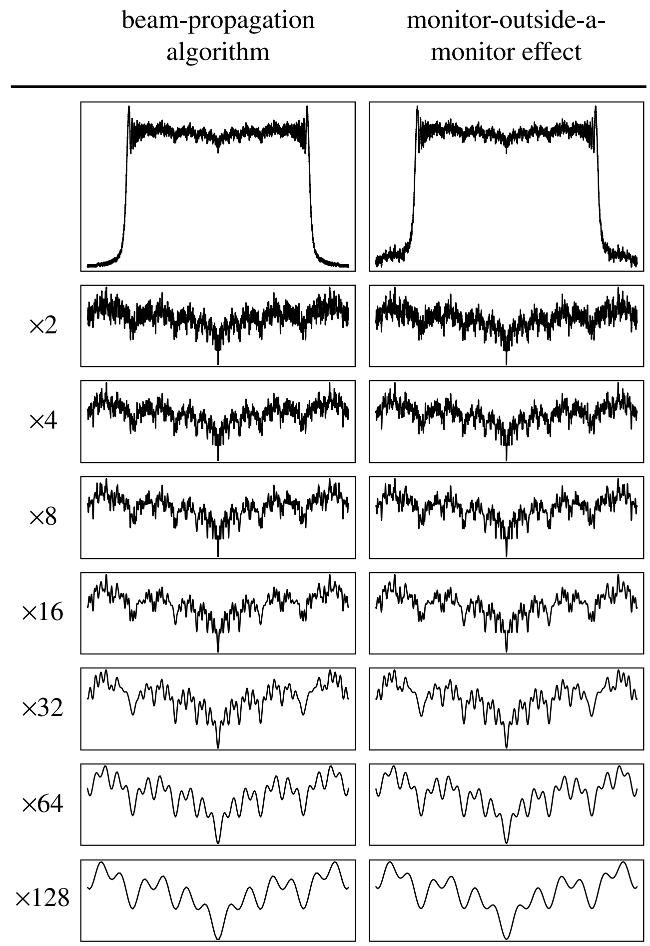

FIG. 6. Center of the amplitude distribution in the plane S_M of the eigenmode of the resonator from Fig. 5, shown with different magnifications and calculated using two different algorithms. The top graphs correspond to a width of 15 mm; the patterns shown for magnification $\times 32$ are the same as those shown for $n = 20$ in Fig. 5. Note the similarity of the cross section under different magnifications, which demonstrates the diffraction-limited self-similarity of the pattern.

approximated as the result of n iterations of the monitor-outside-a-monitor effect with magnification M and pixel function $p(x)$.

A number of more subtle points still have to be taken into account. First, the new structure part is stated to be the sum of $p(x)$ and $p(x)$, stretched by a factor M; this is true only if the phases of these two parts are the same (so that their amplitudes simply add up), which, to a good approximation, is the case for beams with the eigenmode's phase structure. Consequently the pixel function $p(x)$ is the diffraction pattern resulting from a uniform amplitude distribution in the plane S_M with the eigenmode's phase distribution. Second, in order to conserve power in the beam, as both parts of the amplitude distribution get stretched by a factor M, their individual amplitudes are reduced by a factor of $1/\sqrt{|M|}$. The structure part of the amplitude distribution after n round-trips can be written as

$$s_n(x) \approx \sum_{i=0}^{n-1} \frac{p(\frac{x}{M^i})}{\sqrt{|M|^n}} \,. \tag{2}$$

Obviously, as this mechanism does not take into account the clipping by the aperture of the stretched structural part, Eq. (2) is valid only for the central part of the beam that has not been clipped by the aperture.

To evaluate the accuracy of this proposed mechanism, we compare the amplitude distributions calculated with the help of the monitor-outside-a-monitor effect to those calculated with a traditional beam-propagation method. We initialize the beam-propagation method with a beam cross section in the plane S_M of uniform amplitude and the phase structure of the resonator's eigenmode and trace the corresponding beam through the resonator, using a Fourier algorithm [12] to model free-space propagation and multiplication with appropriate phase factors to represent the effect of lenses [13]. An array of 32 768 complex numbers was used to represent the amplitude of a light beam of wavelength $\lambda = 633$ nm over a physical width of 30 mm. The calculated amplitude distribution in the plane S_M after one round-trip was taken as the pixel function $p(x)$ for the monitor-outside-a-monitor effect. Figures 5 and 6 show the detailed structure after $n = 1, 2, 3, \ldots$ round-trips and the structure under magnification of the eigenmode of a confocal resonator, respectively; Fig. 7 shows the detailed structure in the eigenmodes of some other resonators. In

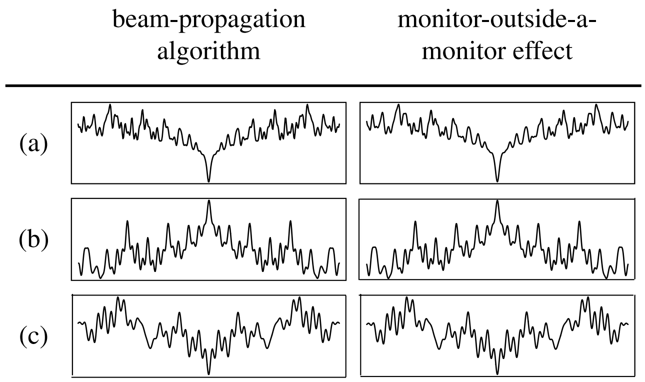

FIG. 7. Amplitude cross sections in the plane S_M through the eigenmodes of various resonators, calculated using a beam-propagation algorithm and the monitor-outside-a-monitor effect. (a) Confocal resonator, $f_1 = 20$ mm, $f_2 = 30$ mm (so $M = -1.5$), $r = 3.56$ mm; (b) confocal resonator, $f_1 = 20$ mm, $f_2 = 40$ mm (so $M = -2$), $r = 2.76$ mm; (c) nonconfocal resonator, left mirror with focal length $f = 20$ mm and aperture radius $r = 2.76$ mm; plane right mirror without aperture, separation between mirrors 53.33 mm, leading to $M = -3$.

every case, the agreement between the output from the two methods is so good that they are hard to distinguish.

The reader should be aware of an important limitation of our mechanism. One of the assumptions of this mechanism is that the structural part of a light beam is simply stretched (and clipped at the edges) during a round-trip through the resonator. To a good approximation, this assumption is justified in resonators with high Fresnel numbers, as in the examples presented in Figs. 5 and 6, all of which were calculated for $N \approx 100$. However, the assumption is not justified in resonators with lower Fresnel numbers ($N \approx 1$); as a consequence, the monitor-outside-a-monitor effect yields inaccurate results for such resonators.

Clearly the monitor-outside-a-monitor effect explains in detail the eigenmode structure in the self-conjugate plane S_M. How about the other, non-self-conjugate, planes? These planes are not geometrically imaged onto themselves, and it is during this imaging that the important stretching of the beam's structure occurs. On the other hand, a basic ray-optical picture of an unstable resonator's eigenmode, in which every light ray passes through the high-intensity spot in the plane $S_{1/M}$, predicts stretching by a factor of M of the beam in *every* plane [1,11]. A more careful analysis reveals that the small structure in the beam becomes distorted as well as stretched. As the monitor-outside-a-monitor effect involves perfect stretching, it predicts a pattern with the wrong detail but approximately the correct statistical fractal behavior. It can therefore explain the general fractal character of the eigenmode cross sections reported in unstable resonators [1–3].

IV. Conclusions.—We have proposed a new mechanism for the generation of fractal structure over a limited range of length scales. We call this mechanism the monitor-outside-a-monitor effect.

Although the processes responsible for shaping the eigenmode in unstable resonators are very complex, the simple monitor-outside-a-monitor effect predicts fractal eigenmode structure in particular planes very accurately.

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