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Hierarchy of Local Minimum Solutions of Heisenberg's Uncertainty Principle

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We derive a new hierarchy of local minimum Heisenberg-uncertainty states by introducing a superposition of "small waves" onto some initial state. Our objective is to increase the resolution in one observable, with the least decrease in the resolution in the conjugate observable. This leads to a constrained minimization which in a well-defined sense yields the best possible way of achieving this goal. The results are relevant to many topics (e.g., quantum optics and control, Bose-Einstein condensation, path integration, etc.).

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I. Introduction.— At the heart of the difference between quantum and classical mechanics is the noncommutativity of conjugate observables, which leads to the Heisenberg uncertainty principle [1,2]. It states that one cannot specify, simultaneously, exact values (eigenvalues) of a pair of noncommuting observables (e.g., position and momentum) and places quantitative restrictions on their relative variances. The essential origin of this principle is that quantum mechanics possesses the mathematical structure of a linear vector space (viz., a Hilbert space) [3]. The transformation between, e.g., the position representation of the vector space and the canonically conjugate momentum representation is simply a Fourier transform [3]. The usual derivation of the uncertainty principle makes use of Schwarz's inequality [1,2], yielding

$$\langle\psi|\hat{x}^2|\psi\rangle\langle\psi|\hat{p}_x^2|\psi\rangle \geq |\langle\psi|\hat{x}\hat{p}_x|\psi\rangle|^2. \qquad (1)$$

Here $\langle\psi\,|\,\psi\rangle = 1$, and $\hat{x}$ and $\hat{p}_x$ are conjugate position and momentum operators, with zero expectation values assumed for convenience. (Nonzero expectation values are discussed elsewhere [4,5].) Relating $\langle\psi|\hat{x}\hat{p}_x|\psi\rangle$ to the average of the commutator $[\hat{x}, \hat{p}_x]$ is a textbook exercise [1]. The equality is satisfied when

$$\left(\frac{\hat{x}}{\sigma} + \frac{i\sigma\hat{p}_x}{\hbar}\right)|\psi\rangle = \left(\frac{\hat{x}}{\sigma} + i\sigma\hat{k}\right)|\psi\rangle = 0\,, \qquad (2)$$

and the solution of this equation has a minimum uncertainty product. Here σ, in general, is complex [1,2], $\hbar\hat{k} = \hat{p}_x$, and $\frac{\hat{x}}{\sigma}$ and $\frac{\sigma\hat{p}_x}{\hbar}$ are dimensionless. Obviously $(\frac{\hat{x}}{\sigma} + \frac{i\sigma\hat{p}_x}{\hbar}) \equiv (\frac{\hat{x}}{\sigma} + i\sigma\hat{k})$ is an annihilation operator and the solution to Eq. (2) is the vacuum state [2,6], i.e., a Gaussian in *both* the position and the momentum representations. Though there are *many* specific realizations of the Gaussian, e.g., coherent states and squeezed states, *all* are captured by a single function (of a dimensionless variable, $\xi \equiv k^2\sigma^2/2$) solving

$$i\,\frac{d\psi}{dk} + i\sigma^2 k\psi = \frac{d\psi}{d\xi} + \psi = 0\,. \qquad (3)$$

All Gaussians are then members of a family of "corresponding states," with different σ values but the same uncertainty.

Now all phenomena governed by similar mathematics will be subject to some form of uncertainty principle, and the range of such phenomena is vast [7–23]. The uncertainty principle imposes fundamental limitations on the "bandwidth" and "duration" of signals used to manipulate, transfer, capture, extract, etc., information, and is thus relevant in many important technologies. One example is quantum optics [9,17–23]. Here attention centers on Gaussian light pulses with (1) the minimum uncertainty product and (2) a squeezed variance in one or the other

observable so that the uncertainty is not equally shared. The general problem of localizing photon packets in both physical and Fourier spaces has yet to be solved completely.

The mathematics of Fourier and harmonic analysis also underlie signal processing and communications, and the general problems of constructing and analyzing signals that are sufficiently sharply defined, both in physical and Fourier domains, are of fundamental importance. Major efforts over the past 15 years have focused on the so-called "little wave" or "wavelet" revolution [11,24–27]. The basic idea is to "filter" the usual plane waves of Fourier analysis, $\exp(ikx)$, by weighting them with some localized function. Usually the weight function has rapid decay in both the x and k domains. A widely used example is the Gaussian weight; one realization is the coherent state [28]. This decomposes a signal (state) into separate pieces which are themselves (approximately) localized in k and x. The signal (state) is typically localized into separate low bandpass and one or more high bandpass chunks, each of whose Fourier transform is also localized. Alternatively, the full signal (state) can be generated by an appropriate superposition of such chunks (i.e., the procedure can be carried out in either direction). This idea of parsing a signal into such chunks is called a "multiresolution analysis." (Our discussion has, perforce, been qualitative; rigorous formulations abound in the mathematics literature [7,8,11–13,24–27].) Of course, the Heisenberg uncertainty principle applies throughout, so that if one increases the x localization by superposing some initial wave ψ, with another small wave or wavelet, $\delta\psi$, then typically the resulting new wave is more delocalized in k. A fundamental question is how can this process be optimized? The objective of this paper is to show how one can obtain a multiresolution realization of Heisenberg's uncertainty. We shall see that in doing this, we obtain a new hierarchy of locally minimal-uncertainty states. Of course, one approach to the problem of "squeezing," say, x at the cost of delocalizing in k is via the Gaussian. However, this limits the overall pulse or signal (state) to a fixed form.

The fundamental issue, thus, reduces to the following question: Is there a realization of the uncertainty principle incorporating multiresolution? In another paper, we discuss the following intimately related question: Is there a generalization of the Gaussian family of functions [5]? *The relationship between these two questions rests on the point of view that Eq. (2) serves to define Gaussians [1,2].*

This paper is organized as follows. We consider the analysis by which the minimum uncertainty is derived and show how to combine it with the multiresolution idea. This leads to a hierarchy of local-minimum uncertainty distributed approximating functions and a related hierarchy of "μ wavelets," whose properties are briefly discussed. It is shown that the absolute minimum Heisenberg-uncertainty state results from an appropriate limit of the hierarchy obtained using the local-minimum uncertainty condition.

II. Combining the uncertainty and multiresolution conditions.—For reasons that will become apparent, it is convenient to consider state vectors that are not normalized in the usual sense, but instead satisfy the condition that $\langle k \mid \psi\rangle$ equals one at the k value of zero. Then the unit normalized ket vector is $|\psi\rangle/N$, $N^2 = \langle\psi \mid \psi\rangle$. Since the momentum representation wave function, $\langle k \mid \psi\rangle$, obeying the above condition, is dimensionless, it depends only on dimensionless variable combinations. We restrict our consideration to the case where there is only a single parameter, σ, with dimensions of length, so in the wave number representation the wave function depends only on the dimensionless combination $k\sigma$. As a function of k, the wave functions generated by different values of σ form a family of corresponding states. We now modify a given initial state vector to create a new state,

$$|\psi_n\rangle = |\psi_{n-1}\rangle + |\delta\psi_n\rangle, \tag{4}$$

where $|\delta\psi_n\rangle$ denotes the modifying vector. Initially $|\delta\psi_n\rangle$ is subject only to two conditions: (a) *that it satisfies an "admissibility" requirement [25], [26]*, and (b) *that it incrementally increases the x (or k, if one so chooses) resolution of* $|\psi_{n-1}\rangle$. The first condition states that

$$\langle k \mid \delta\psi_n\rangle = 0, \qquad k = 0 \tag{5}$$

(guaranteeing that $\langle k \mid \psi_n\rangle = \langle k \mid \psi_{n-1}\rangle = 1$ at $k = 0$). Thus,

$$\int_{-\infty}^{\infty} dx \langle x \mid \delta\psi_n\rangle = 0\,. \tag{6}$$

This condition provides an intuitive, commonly used definition of a wavelet (little wave) [8]; we shall denote our $|\delta\psi_n\rangle$ as a wavelet. However, see Ref. [26] for a precise discussion of admissibility and wavelets. The second condition usually results in an increase in the uncertainty product. However, for certain choices of initial state, it is possible to increase the resolution in x, while at the same time decreasing the overall uncertainty product.

The uncertainty product for x and k in state $|\psi_n\rangle$ is given by

$$\Delta_n^2 = (\Delta x)_n^2 (\Delta k)_n^2\,, \tag{7}$$

where

$$(\Delta x)_n^2 = \frac{\langle\psi_n|\hat{x}^2|\psi_n\rangle}{N_n^2}, \qquad (\Delta k)_n^2 = \frac{\langle\psi_n|\hat{k}^2|\psi_n\rangle}{N_n^2}\,. \tag{8}$$

Note that varying $|\delta\psi_n\rangle$ to minimize Δ_n^2 subject to Eq. (5) merely forces $|\psi_n\rangle$ to be the absolute minimum uncertainty state. Thus, $\langle k \mid \delta\psi_n\rangle$ given by $\exp(-\xi) - \langle k \mid \psi_{n-1}\rangle$ satisfies Eq. (5), but for this choice $\langle k \mid \psi_n\rangle$ is *identically* the minimum-uncertainty Gaussian. Our goal is to squeeze the x variance of the initial state by minimally increasing its k variance, *not to make a total replacement of* $|\psi_{n-1}\rangle$. *To accomplish our aim requires a modification of the variational procedure, which will allow us to identify local minima consistent with condition (b).*

To squeeze the dispersion in x at the expense of broadening it in k, we begin by writing the uncertainty product in the nonsymmetric form

$$\Delta_n^2 = (\Delta x)_n^2\{[(\Delta k)_n^2]_0 + [(\Delta k)_n^2]_1\}, \tag{9}$$

where

$$[(\Delta k)_n^2]_0 = N_n^{-2}\{\langle\psi_{n-1}|\hat{k}^2|\psi_{n-1}\rangle + 2\,\mathrm{Re}\langle\psi_{n-1}|\hat{k}^2|\delta\psi_n\rangle\} \tag{10}$$

and

$$[(\Delta k)_n^2]_1 = N_n^{-2}\langle\delta\psi_n|\hat{k}^2|\delta\psi_n\rangle. \tag{11}$$

We could reverse the roles of x and k if we should so choose. The direct term, $[(\Delta k)_n^2]_1$, in the k variance is never negative and therefore gives an intrinsically positive contribution. It results from effectively extending the frequency range of the state. The term $[(\Delta k)_n^2]_0$ (which is *not* necessarily positive) contains the remainder of the k variance.

In order to guarantee satisfaction of condition (b) (requiring that the wavelet incrementally increase resolution), we hold the quantity $[\Delta_n^2]_0 \equiv (\Delta x)_n^2[(\Delta k)_n^2]_0$ fixed. Since $[(\Delta k)_n^2]_1$ is non-negative, according to Eq. (9), the fixed value of $[\Delta_n^2]_0$ provides a floor under which the uncertainty product cannot go. The value of $[\Delta_n^2]_0$ is not known *a priori*, but will be subsequently determined. We seek to minimize the uncertainty product with respect to variations of the wavelet, $|\delta\psi_n\rangle/N_n$, by minimizing $[(\Delta k)_n^2]$. The choice of $[\Delta_n^2]_0$ parametrizes a family of constrained minima; the state of absolute minimum uncertainty lies within this family. Using the Schwarz inequality,

$$\Delta_n^2 \geq [\Delta_n^2]_0 + \left|\left\{\frac{1}{N_n}\langle\psi_n\,|\,\hat{x}\}\{\hat{k}\,|\,\delta\psi_n\rangle\frac{1}{N_n}\right\}\right|^2. \tag{12}$$

Except for the trivial case $|\delta\psi_n\rangle \equiv 0$, the equality in Eq. (12) clearly holds only when $\hat{x}|\psi_n\rangle/N_n$ is proportional to $\hat{k}|\delta\psi_n\rangle/N_n$. This gives constrained local minima consistent with condition (b) (however, not all minima satisfy this relationship; see [4] for an expanded discussion). In the k representation the resulting equation is

$$-\frac{1}{(\sigma')^2}\frac{\partial\psi_n}{\partial k} = k(\psi_n - \psi_{n-1}) \equiv k\phi_n, \tag{13}$$

where σ' is a constant and ϕ_n is the k representation of the wavelet. It is apparent that if ψ_{n-1} obeys a law of corresponding states in terms of the dimensionless variable $\xi \equiv \frac{k^2\sigma^2}{2}$, then so also will ψ_n *if and only if* $\sigma' = \sigma$. Equation (13) then becomes

$$-\frac{\partial\psi_n}{\partial\xi} = \phi_n. \tag{14}$$

Solutions of this equation are determined by the constant of integration and ψ_{n-1}, which in turn uniquely determine ϕ_n. In contrast to Eq. (2), this equation gives rise to *local relative minimum uncertainties, constrained by the floor,* $(\Delta_n^2)_0$. The *general* solution to this linear, first order, inhomogeneous differential equation is

$$\psi_n(\xi) = \beta e^{-\xi} + \int_0^\xi d\xi'\, e^{-(\xi-\xi')}\psi_{n-1}(\xi'), \tag{15}$$

where β is the constant of integration, given by

$$\beta = \psi_n(\xi = 0). \tag{16}$$

Clearly for $\lim_{\xi\to\infty}\psi_n(\xi) = 0$ to hold, we must require that $\mathrm{Re}(\sigma^2) > 0$. However, subject to this, the constant σ^2 can be complex. In this paper we will deal only with real σ^2 [4].

When there is only one dimensionless variable, it is straightforward to show that Δ_n^2 is independent of σ because x and k are conjugate variables. *However, Δ_n^2 does depend on β, as we now consider.* First, it is easily proved that the normalized state vector $|\psi_n\rangle/N_n$ behaves according to

$$\lim_{\beta\to\infty}\frac{\langle k\,|\,\psi_n\rangle}{N_n} = \frac{\sigma}{\sqrt{2\pi}}e^{-\sigma^2k^2/2} \tag{17}$$

in the infinite-β limit ($\psi_{n-1} \ll \delta\psi_n$), and $[\Delta_n^2]_0 = 0$. That is, in the general case when $\beta \gg 1$, the relative minimum uncertainty state $\psi_n(k)$ becomes a Gaussian. Thus, the absolute minimum uncertainty state is pushed to the $\beta \to \infty$ limit by the reformulation to a constrained minimization. However, finite values of β give rise to states of *relative minimum uncertainty,* provided $\psi_{n-1}(\xi) \neq 0$. [If $\psi_{n-1}(\xi) = 0$, Eqs. (14) and (3) are identical, and satisfied by the Gaussian.]

We now impose the admissibility requirement, condition (a). According to Eq. (15) this requires that we set $\beta \equiv 1$ *independent of n* to obtain

$$\psi_n(\xi) = e^{-\xi} + \int_0^\xi d\xi'\, e^{-(\xi-\xi')}\psi_{n-1}(\xi'). \tag{18}$$

For a given ψ_{n-1}, this uniquely fixes $[\Delta_n^2]_0$. The extent to which this minimum-uncertainty procedure improves resolution depends on $\psi_{n-1}(\xi)$. A natural starting point, in analogy to the eigenvectors of the annihilation operator, is to begin with $\psi_{-1}(\xi) \equiv 0$; *then Eq. (18) uniquely yields the Hermite distributed approximating functionals (HDAF)* [29–35]:

$$\psi_n(\xi) = e^{-\xi}\sum_{j=0}^{n}\frac{\xi^j}{j!}, \tag{19}$$

or, replacing ξ by $\sigma^2k^2/2$,

$$\psi_n(k) = e^{-\sigma^2k^2/2}\sum_{j=0}^{n}\left(\frac{\sigma^2k^2}{2}\right)^j/j!. \tag{20}$$

We note that the starting HDAF, $\psi_0(\xi)$, is the absolute minimum uncertainty Gaussian. The sense of the optimization is that $\psi_n(k)$ gives the smallest possible increase in the variance Δk when Δx is squeezed from its $(n-1)$ value by adding $\exp(-\frac{\sigma^2k^2}{2})(\frac{\sigma^2k^2}{2})^n/n!$ to the

HDAF $\psi_{n-1}(k)$ for a value of $[\Delta_n^2]_0$ determined by the admissibility requirement.

Starting with an arbitrary initial state, $\psi_0(\xi)$, we can build up a more highly squeezed state sequentially by adding μ wavelets, where each addition is done with minimum uncertainty as discussed. We refer to the set of wavelets thus generated as a *wavelet family*. It is readily seen that each member of such a family obeys [Eq. (13)]

$$-\frac{\partial \phi_n}{\partial \xi} = \phi_n - \phi_{n-1}, \tag{21}$$

which has the solution

$$\phi_n(\xi) = \int_0^\xi d\xi' \, e^{-(\xi-\xi')} \phi_{n-1}(\xi'), \tag{22}$$

obeying the admissibility requirement. It should be pointed out that while the uncertainty at each step is independent of σ, it does depend on n. As we have discussed, this must be the case if the μ wavelets are to generate a multiresolution hierarchy. Integrating Eq. (14) and making use of the fact that $\psi_n(\xi = 0) = 1$, we have that

$$c_n \equiv \int_0^\infty d\xi \, \phi_n(\xi) = 1 \tag{23}$$

and hence c_n is independent of n for every member of the wavelet family. For the HDAFs of Eq. (20), the μ wavelets are

$$\phi_n(\xi) = e^{-\xi} \frac{\xi^n}{n!} = \frac{e^{-(\sigma^2 k^2)/2}}{n!} \left(\frac{\sigma^2 k^2}{2}\right)^n. \tag{24}$$

The asymptotic (large n) behavior of the μ wavelets is determined by a "diffusion equation" for a source traveling at a constant "velocity,"

$$\frac{\partial \phi_n}{\partial n} = -\frac{\partial \phi_n}{\partial \xi} + \frac{1}{2} \frac{\partial^2 \phi_n}{\partial \xi^2}, \tag{25}$$

where n plays a "timelike" role [4]. For large n,

$$\phi_n(\xi) \to \frac{1}{\sqrt{2\pi n}} e^{-(\xi-n)^2/2n}, \tag{26}$$

which is the diffusion profile of an initial delta-function source. It can also be shown from this result that the large n profiles of the leading edges of ψ_n as a function of k are independent of n [4]. Finally, the large n asymptotic dependence of the uncertainty product Δ_n scales like $n^{1/4}$. This compares, for example, to the harmonic oscillator states for which $\Delta_{\mathrm{HO},n} \equiv n + \frac{1}{2}$. The uncertainty product of the $n > 0$ HDAF states is smaller than for the corresponding harmonic oscillator states [35].

In conclusion, we expect the μ wavelets and HDAFs to be useful for experimentally creating new, optimal squeezed states. Such packets should be of use in quantum optics applications and for Bose-Einstein condensates. One specific possibility is the creation of light pulses that are as close as possible to an ideal square pulse. This could be useful for communications and for high precision measurements. Additionally, the ability to shape coherent light pulses more precisely may be of use in quantum computing. Another application could be for optimally confining atomic and molecular systems at low temperatures.

Just as coherent states are useful for semiclassical dynamics, by making use of their dependence on the continuously variable complex eigenvalues of the annihilation operator, so also HDAFs and μ wavelets vary with the analogous parameter, σ. It may be possible to use HDAFs to write path integrals in phase space and obtain new semiclassical expressions that contain the parameter n. These and other areas of potential application are currently under study.

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