Casimir Force at Both Nonzero Temperature and Finite Conductivity

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We find the combined effect of nonzero temperature and finite conductivity onto the Casimir force between real metals. Configurations of two parallel plates and a sphere (lens) above a plate are considered. Perturbation theory in two parameters (the relative temperature and the relative penetration depth of zero-point oscillations into the metal) is developed. Perturbative results are compared with computations. Recent improper computations based on the Lifshitz formula for the temperature Casimir force are discussed.

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The Casimir effect is interesting because it is caused by the zero-point oscillations of quantized fields. An enormous amount of theoretical and experimental development has recently become available. Theoretical progress was made in elaborating different approximate methods [1–3] and in the problem of a dielectric sphere where, for instance, the structure of the ultraviolet divergencies had been clarified [4]. In [5] the additive method was applied to a dilute dielectric ball, and in [6] progress was made in obtaining analytical results. The Casimir force was demonstrated between metallic surfaces of a sphere above a disk using a torsion pendulum [7] and an atomic force microscope [8,9].

The increased accuracy of Casimir force measurements invites a further investigation of different theoretical corrections. In [10] the Casimir force for the configuration of a sphere above a plate was computed by taking into account surface roughness and finite conductivity corrections. That result is in excellent agreement with the measured Casimir force. Except for contributions of surface roughness and finite conductivity, corrections due to nonzero temperature play a dominant role above some distance between the test bodies. The general expression for the temperature Casimir force between dielectric plates was first obtained by Lifshitz [11] (see also [12]). The Lifshitz result is generally used in the theoretical predictions. In this paper we address modifications to the Lifshitz result which are necessary to correctly take into account the combined effect of the finite temperature and finite conductivity. The temperature Casimir force between perfectly conducting plates was found in [13–15], including the limiting cases of large and small plate separations (high and low temperatures). These results were modified for the configuration of a spherical lens above a disk in [7]. The temperature corrections are found to be insignificant within the separations of experiments [8,9] (from $a \approx 0.1$ μm until $a = 0.9$ μm or 0.5 μm). As for experiment [7] they constitute up to 174% of the net force at the largest separation $a = 6$ μm [16].

Computations of the recent papers [17–19] were based on the Lifshitz formula for the van der Waals and Casimir force at nonzero temperature. Drude and plasma models were used to represent the dependence of dielectric permittivity along the imaginary axis in the plane of complex frequency. As discussed below, the ambiguity contained in the zero frequency term of the Lifshitz formula may lead to incorrect computational results, including the occurrence of large temperature corrections at small separations and wrong asymptotics at high temperatures.

Here we present a perturbative calculation of the combined influence of nonzero temperature and finite conductivity on the Casimir force. The obtained results are the generalization of [13–15] to the case of real metals. They can be used for the interpretation of precision experiments on Casimir force. No unexpected large temperature contributions arise at small separations. The more correct representation for the Lifshitz formula is discussed avoiding the ambiguity at zero frequency and giving the possibility to eliminate the defects of [17–19].

We start with the configuration of two plane parallel plates with the dielectric permittivity ε separated by an empty gap of thickness a. At arbitrary temperature T the attractive force per unit area acting between plates is given by the Lifshitz formula [12]

$$F_{pp}(a) = -\frac{k_B T}{8\pi a^3} \sum_{n=0}^{\infty}{}' \varphi_{pp}(x_n), \qquad \varphi_{pp}(x_n) = \int_{x_n}^{\infty} z^2\, dz\, (Q_1^{-1} + Q_2^{-1}), \tag{1}$$

where

$$Q_{1,2} = r_{1,2}^{-2} e^z - 1, \qquad r_1 = \frac{\sqrt{x_n^2(\varepsilon - 1) + z^2} - \varepsilon z}{\sqrt{x_n^2(\varepsilon - 1) + z^2} + \varepsilon z}, \qquad r_2 = \frac{\sqrt{x_n^2(\varepsilon - 1) + z^2} - z}{\sqrt{x_n^2(\varepsilon - 1) + z^2} + z}, \qquad \varepsilon \equiv \varepsilon(i x_n). \tag{2}$$

Here $r_{1,2}$ are the reflection coefficients for the waves with different polarization; the prime on the sum indicates that the term with $n = 0$ is to be taken with the coefficient $1/2$, $x_n = 2a\xi_n/c \equiv \tau n \equiv 2\pi n T/T_{\text{eff}}$, where the effective temperature is defined by $k_B T_{\text{eff}} = \hbar c/(2a)$ [20].

The sum in (1) can be calculated with the help of the Abel-Plana formula [20]. The result is

$$F_{pp}(a) = -\frac{k_B T}{8\pi a^3}\left[\frac{1}{\tau}\int_0^\infty dx \int_x^\infty z^2\, dz (Q_1^{-1} + Q_2^{-1}) + i\int_0^\infty \frac{\varphi_{pp}(i\tau y) - \varphi_{pp}(-i\tau y)}{e^{2\pi y} - 1}\, dy\right]. \quad (3)$$

The first term in the right-hand side of (3) is the Casimir force at zero temperature, the second one takes into account the temperature corrections. The zero temperature contribution was calculated in [21] numerically by the use of optical tabulated data for the complex refractive index (an alternative computation [22] contains some errors which are indicated in [21]). Independently, in [23] it was determined by perturbation theory up to the fourth order in the small parameter δ_0/a (δ_0 being the effective penetration depth of electromagnetic zero point oscillations into the metal). Thereby, the plasma model was used for the dielectric permittivity

$$\varepsilon(i\xi) = 1 + \frac{\omega_p^2}{\xi^2} = 1 + \frac{\tilde{\omega}_p^2}{x^2}, \quad (4)$$

where ω_p is the effective plasma frequency, and $\tilde{\omega}_p = 2a\omega_p/c$, so that $\alpha \equiv 1/\tilde{\omega}_p = \delta_0/(2a)$. The results of [21] and [23] are in good agreement for space separations $a \geq \lambda_p = 2\pi c/\omega_p$. It is well known that the plasma model does not take into account the contribution of relaxation processes which are taken into consideration by the Drude model (see below). However, the variation of the Casimir force obtained by both models remains smaller than 2% [21].

Let us calculate the second term of (3) in the application range of plasma model and under the condition $T \ll T_{\text{eff}}$. To do this we use the representation of (1)

$$\varphi_{pp}(x) = \varphi_{pp}^{(1)}(x) + \varphi_{pp}^{(2)}(x) \equiv \int_x^\infty z^2\, dz\, Q_1^{-1} + \int_x^\infty z^2\, dz\, Q_2^{-1}. \quad (5)$$

Introducing $x_0 < 1$ one can write

$$\varphi_{pp}^{(i)}(x) = \int_x^{x_0} z^2\, dz\, Q_i^{-1} + \int_{x_0}^\infty z^2\, dz\, Q_i^{-1}. \quad (6)$$

Considering first the case $i = 2$, we notice that in the plasma model (4) the second term from the right-hand side of (6) does not depend on x. Expanding Q_2^{-1} from the first term of (6) into a series in powers of z and integrating one arrives at the result

$$\varphi_{pp}^{(2)}(x) = C - \frac{1}{1+4\alpha}\frac{x^2}{2} + \frac{x^3}{6} - \frac{1 + 16\alpha + 96\alpha^2 + 264\alpha^3 + 288\alpha^4}{12(1+4\alpha)^3}\frac{x^4}{4} + O(x^6), \quad (7)$$

where $C = \text{const}$.

The even powers of x evidently do not contribute to the second term of (3). As a result there is only one temperature correction originating from Q_2 which is caused by the term $x^3/6$ and which does not depend on $\tilde{\omega}_p$. Substituting (7) into the second term of (3) one obtains

$$\Delta_T F_{pp}^{(2)}(a) = F_{pp}^{(0)}(a)\frac{1}{6}\left(\frac{T}{T_{\text{eff}}}\right)^4. \quad (8)$$

Here $F_{pp}^{(0)}(a) = -\pi^2\hbar c/(240a^4)$. Note that we neglect the corrections $O[(T/T_{\text{eff}})^5]$.

Consider now $i = 1$ in (6). In this case both the first and the second terms in the right-hand side depend on x. The second term, however, is an even function of x and for that reason it does not contribute to (3). Let us expand the quantity $z^2 Q_1^{-1}$ in powers of small parameters α and z. Integrating the obtained series between the limits $z = x$ and $z = x_0 < 1$ we arrive to

$$\varphi_{pp}^{(1)}(x) = \frac{x^3}{6} + 4x^2\alpha \ln x + \tilde{\varphi}_{pp}^{(1)}(x), \quad (9)$$

where the quantity $\tilde{\varphi}_{pp}^{(1)}(x)$ contains terms which do not contribute to (3) or lead to contributions of order $(T/T_{\text{eff}})^5$ or higher. Substituting (9) into the second term of (3) we get

$$\Delta_T F_{pp}^{(1)}(a) = F_{pp}^{(0)}(a)\left[\frac{1}{6}\left(\frac{T}{T_{\text{eff}}}\right)^4 + \frac{30\zeta(3)}{\pi^3}\frac{\delta_0}{a}\left(\frac{T}{T_{\text{eff}}}\right)^3\right], \quad (10)$$

where $\zeta(3) \approx 1.202$ is the Riemann zeta function.

Now, let us take together (8), (10) and the zero temperature contribution given by the first term of (3). In [23] the last one was calculated up to the fourth order. Here we add two more orders. The final result is

$$F_{pp}(a) = F_{pp}^{(0)}(a)\left\{1 + \frac{1}{3}\left(\frac{T}{T_{\text{eff}}}\right)^4 - \frac{16}{3}\frac{\delta_0}{a}\left[1 - \frac{45\zeta(3)}{8\pi^3}\left(\frac{T}{T_{\text{eff}}}\right)^3\right] + \sum_{i=2}^{6} c_i \frac{\delta_0^i}{a^i}\right\}, \quad (11)$$

where $c_2 = 24$, and the other coefficients are

$$c_3 = -\frac{640}{7}\left(1 - \frac{\pi^2}{210}\right), \qquad c_4 = \frac{2800}{9}\left(1 - \frac{163\pi^2}{7350}\right),$$
$$c_5 = -\frac{10752}{11}\left(1 - \frac{305\pi^2}{5292} + \frac{379\pi^4}{1693440}\right), \tag{12}$$
$$c_6 = \frac{37632}{13}\left(1 - \frac{1135\pi^2}{9720} + \frac{2879\pi^4}{1358280}\right).$$

For $\delta_0 = 0$ (perfect conductor) Eq. (11) turns into the well-known result [13–15]. It is significant that the first correction of mixing finite conductivity and finite temperature is of order $(T/T_{\rm eff})^3$, and there are no temperature corrections up to $(T/T_{\rm eff})^4$ in the higher conductivity corrections from the second up to the sixth order.

Analogous calculations can be performed for the configuration of a sphere (lens) of radius R above a plate starting from the force

$$F_{\rm pl}(a) = \frac{k_B TR}{4a^2} {\sum_{n=0}^{\infty}}' \int_{x_n}^{\infty} z\,dz \ln \frac{Q_1 Q_2}{(Q_1 + 1)(Q_2 + 1)}. \tag{13}$$

This formula is obtained from (1) using the proximity force theorem [24], $Q_{1,2}$ are defined in (2). After straightforward calculations, using [23] for the zero temperature contribution, the result is

$$F_{\rm pl}(a) = F_{\rm pl}^{(0)}(a)\left\{1 + \frac{45\zeta(3)}{\pi^3}\left(\frac{T}{T_{\rm eff}}\right)^3 - \left(\frac{T}{T_{\rm eff}}\right)^4 - 4\frac{\delta_0}{a}\left[1 - \frac{45\zeta(3)}{2\pi^3}\left(\frac{T}{T_{\rm eff}}\right)^3 + \left(\frac{T}{T_{\rm eff}}\right)^4\right] + \sum_{i=2}^{6}\tilde{c}_i\frac{\delta_0^i}{a^i}\right\}, \tag{14}$$

where $F_{\rm pl}^{(0)}(a) = -\pi^3\hbar cR/(360a^3)$, $\tilde{c}_i = 3c_i/(3 + i)$, c_i are defined in (12). For the perfect conductor $\delta_0 \to 0$ the known asymptotic behavior [7] is reproduced.

Now we consider space separations a for which $T \sim T_{\rm eff}$ or even larger. In this case perturbation theory in $T/T_{\rm eff}$ does not work. Let us compute the values of temperature force (13) numerically in dependence on a for Al surfaces used in experiments [8,9] with $\omega_p = 1.92 \times 10^{16}$ rad/s [25], $T = 300$ K, and $R = 100\ \mu$m. The numerical results are shown in Fig. 1 by the solid curve. In the same figure the asymptotic behavior (14) is presented by the dotted line. The dashed line shows the Casimir force at zero temperature (but with account of finite conductivity). Here, the force was computed by Eq. (13) in which the sum has been changed into the integral [15]. It is seen that perturbation theory works well within the range $0.1\ \mu\text{m} \le a \le 3.5\ \mu$m (note that all six perturbation orders are essential near the left verge of this interval). Starting from $a = 6\ \mu$m the solid line represents the asymptotics at large separations (temperatures)

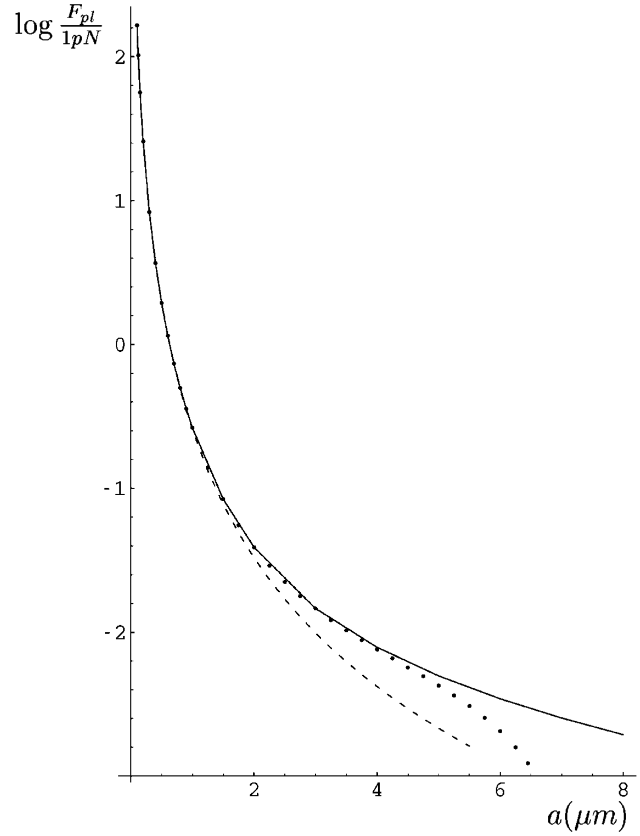

FIG. 1. The Casimir force as a function of the surface separation in configuration of a sphere above a disk. The solid line represents the computational results obtained by Eqs. (4) and (13). The dotted line is calculated by the perturbative Eq. (14). The dashed line is the zero temperature result.

$$F_{\rm pl}(a) = -\frac{\zeta(3)}{4a^2} Rk_B T\left(1 - 2\frac{\delta_0}{a}\right). \tag{15}$$

This result follows from the term of (13) with $n = 0$ (the other terms being exponentially small in $T/T_{\rm eff}$). For $\delta_0 = 0$ one obtains from (15) the known expression for perfect conductors [7,13–15]. Finite conductivity corrections of higher orders do not contribute at large separations.

In [17–19] the Drude model was used also at small frequences for which the dielectric permittivity on the imaginary axis is

$$\varepsilon(i\xi) = 1 + \frac{\omega_p^2}{\xi(\xi + \gamma)}. \tag{16}$$

Here, γ is the relaxation frequency. Substituting Eq. (16) into Eqs. (1) and (2), one obtains the ambiguity in the value of reflection coefficient r_2 at zero frequency. Actually, at $x_0 = 0$ it follows from (2) that $r_2^2 = 1$ for $z = 0$, and $r_2^2 = 0$ for any $z \neq 0$, i.e., r_2 is discontinuous at the point $z = 0$. Note that if plasma model (4) is used both the coefficients $r_{1,2}$ are continuous. A similar (continuous) behavior of both $r_{1,2}$ holds for the perfect metal if the prescription by J. Schwinger, L. L. DeRaad, Jr., and K. A. Milton [15] is adopted (the limit $\varepsilon \to \infty$ should be taken before setting $\xi = 0$). In [17–19] no account has been taken to the discontinuity of r_2 mentioned above. In [17,18] the value of $r_2 = 0$ was used at zero frequency (the term Q_2^{-1} does not contribute the force in this case). It is easy to check that this leads to the incorrect asymptotics at large temperatures, i.e., to $-\zeta(3)Rk_BT/(8a^2)$ instead of (15). By contrast, the authors of [19] used the value $r_{1,2}^2 = 1$ at zero frequency both at Drude and plasma models according to the well-known values of reflection coefficients for the real photons. As a consequence, there appears [19] a large linear temperature correction to the Casimir force at small separations which is absent for the perfect metal. The nonphysical photons, however, also contribute to the force. Their reflection coefficient may be not equal to unity since $k^2 \neq 0$, k being photon 4-momentum.

It is common knowledge that the Drude model gives an accurate account of the behavior of permittivity at small frequences. The ambiguity in the zero frequency term of Lifshitz formula which arises when one uses the Drude model can be cleared away in the following way. Using the Fourier transformation and Poisson formula, Casimir force (1) can be represented as follows [15,26]:

$$F_{pp}(a) = -\frac{\hbar c}{16\pi^2 a^4} {\sum_{n=0}^{\infty}}' \tilde{\varphi}_{pp}\left(\frac{\hbar c n}{k_B T}\right),$$
$$\tilde{\varphi}_{pp}\left(\frac{\hbar c n}{k_B T}\right) = \int_0^{\infty} dx \cos\left(\frac{n \hbar c x}{2 a k_B T}\right) \varphi_{pp}(x). \quad (17)$$

Here the $n = 0$ term gives the force at zero temperature [which is equal to the first term in the right-hand side of Eq. (3)]. Now the discontinuity of r_2 and Q_2^{-1} at $x = 0$ does not influence the value of $\tilde{\varphi}_{pp}$. That is why the representation (17) of Lifshitz formula is well defined and convenient for computations. It is not difficult to expand $F_{pp}(a)$ from (17) [and also $F_{pl}(a)$] in powers of δ_0/a and to get the coefficients near the first order terms valid at arbitrary temperature. At low temperatures they coincide with (11) and (14), and at high temperatures—with (15). No linear in temperature contributions found in [19] both in plasma and Drude models appear.

In conclusion, it may be said that the combined effect of nonzero temperature and finite conductivity on the Casimir force was examined. The perturbation theory for the force in two small parameters was first developed. Also the ambiguities in the previous calculations were explained and eliminated. The results are in agreement with the previous knowledge for the real metals at zero temperature from one side and for the perfect conductors at nonzero temperature from the other. (Note that some of the above results related to the plasma model only are contained in the independent preprint [26].) The obtained results are the topical ones for the interpretation of precision measurements of the Casimir force.

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