Proposal for Measurement of Harmonic Oscillator Berry Phase in Ion Traps

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We propose a scheme for measuring the Berry phase in the vibrational degree of freedom of a trapped ion. Starting from the ion in a vibrational coherent state we show how to reverse the sign of the coherent state amplitude by using a purely geometric phase. This can then be detected through the internal degrees of freedom of the ion. Our method can be applied to preparation of entangled states of the ion and the vibrational mode.

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When the Hamiltonian of a quantum system is varied adiabatically in a cyclic fashion, the state of the system acquires a geometrical phase in addition to the usual dynamical phase. This effect, discovered by Berry [1] (and generalized in various ways [2]), has been widely tested for two-state systems [3] and attracted interest from a variety of fields [4,5]. However, the Berry phase has not been experimentally measured for quantum harmonic oscillators, though some theoretical calculations exist [6–10]. The reason for this might be the fact that for the simplest case, namely, for adiabatic displacement of an oscillator state in phase space, the Berry phase is independent of the state [6,11] and, thereby, undetectable. However, when a squeezing Hamiltonian is switched on, and the squeezing parameter is varied, there would be a detectable Berry phase [6–8]. For an initial Fock state $|n\rangle$ which undergoes squeezing, the Berry phase after a cycle is $-(n+1/2)$ times the classical Hannay angle [6,7]. The squeezed states of the electromagnetic field would have been a natural candidate to test this kind of phase, but they are not stable enough for an adiabatic evolution. However, the vibrational mode of a trapped ion has been fertile ground for the preparation of long-lived nonclassical states of a harmonic oscillator [12–16]. In this Letter, we derive a Berry phase formula for a certain adiabatic evolution of a joint state of the internal levels of a trapped ion and its vibrational motion. Despite being the phase gained by a joint state, its value is *fundamentally dependent* on the harmonic oscillator nature of the vibrational mode. We propose a scheme to detect this phase which is feasible with current technology. It is worthwhile to mention that, based on the calculations of Ref. [8] (which where tested in systems other than quantum harmonic oscillators [17]), there has been an earlier attempt to detect the *nonadiabatic* geometric phase in ion traps by applying a set of four squeezes to the vibrational state [18]. Here we propose a way of detecting the harmonic oscillator version of Berry's original *adiabatic* geometric phase.

Consider the Hamiltonian

$$H = H_a + H_b\,, \tag{1}$$

where

$$H_a = g_a e^{i\phi}|e\rangle\langle g|a + \text{H.c.}\,, \tag{2}$$

$$H_b = g_b|e\rangle\langle g|a^\dagger + \text{H.c.} \tag{3}$$

In the above, $|e\rangle$ and $|g\rangle$ are two states of a qubit, $a^\dagger$ and a are, respectively, the creation and the annihilation operator of a harmonic oscillator, g_a and g_b are unequal positive interaction strengths (say $g_a > g_b$), and ϕ is an arbitrary phase factor. The motivation for choosing this Hamiltonian is the possibility of its physical implementation, and this will be described later. If the phase ϕ is slowly varied over a complete loop (so that the adiabatic approximation holds true), there will be a nontrivial Berry phase acquired by an eigenstate of the Hamiltonian H. We now proceed to calculate this. We transform the Hamiltonian as

$$H' = S(\epsilon)^\dagger H S(\epsilon)\,, \tag{4}$$

where $S(\epsilon)^\dagger a S(\epsilon) = a\cosh(r) - a^\dagger \sinh(r)e^{i\theta}$ is a squeezing transformation with squeezing parameter $\epsilon = re^{i\theta}$. If we chose the squeezing strength $r = \tanh^{-1} g_b/g_a$ and the squeezing phase $\theta = -\phi$, the transformed Hamiltonian will be

$$H' = \Omega(|e\rangle\langle g|a + a^\dagger|g\rangle\langle e|)\,, \tag{5}$$

where $\Omega = g_a\cosh(r) - g_b\sinh(r)$. H' is the well known resonant Jaynes-Cummings Hamiltonian [19]. A similar transformation to arrive at a Jaynes-Cummings Hamiltonian has been used in Refs. [20,21]. The eigenstates of H' are

$$|\Psi_n^\pm\rangle = \frac{1}{\sqrt{2}}(|g\rangle|n+1\rangle \pm |e\rangle|n\rangle)\,. \tag{6}$$

This implies that the eigenstates of our original Hamiltonian H are

$$|\Psi_n^\pm(\epsilon)\rangle = \frac{S(\epsilon)}{\sqrt{2}}(|g\rangle|n+1\rangle \pm |e\rangle|n\rangle)\,. \tag{7}$$

The states $S(\epsilon)|n\rangle$ featured in the above expression are the squeezed number states [22]. Now we can proceed to calculate the Berry phase from the instantaneous eigenstates $|\Psi_n^\pm(\epsilon)\rangle$ of H.

The expression for the Berry phase for an adiabatic cyclic evolution of a Hamiltonian $H(\mathbf{R})$ in parameter space $\mathbf{R}$ is given in terms of the instantaneous eigenstates $|n, \mathbf{R}\rangle$ of the Hamiltonian as

$$\gamma_n = i \int_c d\mathbf{R} \, \langle n, \mathbf{R} | \nabla_R | n, \mathbf{R} \rangle . \tag{8}$$

Using Eq. (7) in the above equation we get

$$\begin{aligned} \gamma_n &= i \int_c d\epsilon \, \langle \Psi_n^{\pm} | S(\epsilon)^\dagger \nabla_\epsilon S(\epsilon) \, | \Psi_n^{\pm} \rangle \\ &= \frac{i}{2} \int_c d\epsilon \, \langle n + 1 | S(\epsilon)^\dagger \nabla_\epsilon S(\epsilon) \, | n + 1 \rangle \\ &\quad + \frac{i}{2} \int_c d\epsilon \, \langle n | S(\epsilon)^\dagger \nabla_\epsilon S(\epsilon) \, | n \rangle . \end{aligned} \tag{9}$$

If the modulus r of the parameter ϵ is kept constant throughout the evolution, then using the expression for $\langle n | S(\epsilon)^\dagger \nabla_\epsilon S(\epsilon) \, | n \rangle$ from Ref. [6] we get

$$\gamma_n = -2\pi(n + 1) \sinh^2 r . \tag{10}$$

Note that the Berry phase γ_n is the same for both the states $|\Psi_n^+(\epsilon)\rangle$ and $|\Psi_n^-(\epsilon)\rangle$. However, the dynamical phase $\beta_n^{\pm}$ is exactly the opposite for the eigenstates $|\Psi_n^+(\epsilon)\rangle$ and $|\Psi_n^-(\epsilon)\rangle$ and is given by

$$\begin{aligned} \beta_n^{\pm} &= -\int_c dt \, \langle \Psi_n^{\pm} | S(\epsilon)^\dagger H S(\epsilon) \, | \Psi_n^{\pm} \rangle \\ &= \mp \Omega \sqrt{n + 1} \, T , \end{aligned} \tag{11}$$

where T is the time period of the cyclic variation of ϵ. Thus one can make the dynamical phase completely vanish after two cycles by changing the state $|\Psi_n^+(\epsilon)\rangle$ to $|\Psi_n^-(\epsilon)\rangle$ or vice versa after one cycle. Under such circumstances, the only contribution to the phase of the system will be geometrical.

Let us now describe how the Hamiltonian H of Eq. (1) can be physically realized. Recently, ion traps have been a very active field of both experimental [12–15] and theoretical [16,20,21,23] research. Consider a single Λ three-level ion with two hyperfine ground states and an excited state (such as ${}^9\mathrm{Be}^+$ [12]) in a harmonic trap of frequency ν. The two ground states, labeled as $|e\rangle$ and $|g\rangle$, are separated in frequency by an amount ω_0 which is much less than their separation from the excited state. The motion of the ion is modified by its interaction with two pairs of traveling-wave laser beams whose frequencies are detuned from the excited state. The first pair of lasers have frequencies ω_{L1} and ω_{L2}, which satisfy $\omega_{L1} - \omega_{L2} = \omega_0 + \nu$, and the second pair of lasers have frequencies ω_{L3} and ω_{L4}, which satisfy $\omega_{L3} - \omega_{L4} = \omega_0 - \nu$. We require that the pair of lasers $L1$ and $L2$ differ in frequency from the pair $L3$ and $L4$ by an amount much larger than ω_0. We also assume that $L1$ differs in phase from the rest of the beams by an amount ϕ (this phase will need to be slowly varied). The Hamiltonian for this system in the rotating frame with $U = \exp(-i\omega_0\sigma_z t)$ and after making the optical rotating wave approximation is

$$\begin{aligned} H^{(1)} &= \Omega_{12}(e^{i[\phi - \nu t + \eta_{12}(a + a^\dagger)]}\sigma_+ + \mathrm{H.c.}) \\ &\quad + \Omega_{34}(e^{i[\nu t + \eta_{34}(a + a^\dagger)]}\sigma_+ + \mathrm{H.c.}) , \end{aligned} \tag{12}$$

where $a^\dagger$ and a are, respectively, the creation and the annihilation operator for the vibrational modes, $\sigma_- = |g\rangle\langle e|$, $\sigma_+ = |e\rangle\langle g|$, and $\sigma_z = |g\rangle\langle g| - |e\rangle\langle e|$, Ω_{ij} are Rabi frequencies of $|e\rangle \rightarrow |g\rangle$ transition induced by the ith and jth lasers, and the factor $\eta_{ij} = \delta k_{ij} a_0$ is the Lamb-Dicke parameter (where a_0 is the amplitude of the ground state of the trap potential and δk_{ij} is the wave vector difference between the ith and the jth beams). Hamiltonians comprised of any one of the terms $\Omega_{12}(e^{i[\phi - \nu t + \eta_{12}(a + a^\dagger)]}\sigma_+ + \mathrm{H.c.})$ or $\Omega_{34}(e^{i[\nu t + \eta_{34}(a + a^\dagger)]}\sigma_+ + \mathrm{H.c.})$ have been used in the context of nonclassical state preparation of the vibrational mode [12] and in a recent experiment such terms were switched on simultaneously [24]. These are generally implemented with a single pair of traveling wave laser beams. We require two pairs of laser beams to switch on both the Hamiltonian terms simultaneously. Similar Hamiltonian terms, when used simultaneously and in conjunction with atomic decay, can be used to generate squeezed states of motion of the ion [20,21]. Here, however, we do not want atomic spontaneous emission, and thus choose the $|e\rangle$ and $|g\rangle$ to be hyperfine levels so that atomic decay can be neglected. In the Lamb-Dicke limit ($\eta_{ij} \ll 1$) the field can be expanded to the first order in η_{ij}. Expanding thus, and transforming into the rotating frame with $U = \exp(-i\nu a^\dagger a t)$, we obtain

$$\begin{aligned} H^{(1)} = U^\dagger H U &= |e\rangle\langle g| \, (g_a e^{i\phi} a + g_b a^\dagger) + \mathrm{H.c.} \\ &= H_a + H_b , \end{aligned} \tag{13}$$

where we have made another rotating wave approximation by omitting all the rapidly oscillating terms. In the above, $g_a = \eta_{12}\Omega_{12}/2$ and $g_b = \eta_{34}\Omega_{34}/2$. Thus we can obtain the Hamiltonian H in an ion trap by the above methods in a rotating frame. From an experimental perspective, instead of slowly varying the phase of one of the beams relative to the rest, one can alternatively keep the phase of all lasers the same but set $\omega_{L1} - \omega_{L2} = \omega_0 + \nu + \delta$ where $\delta \ll \nu$. Then $t\delta$ which varies very slowly with time t will serve as the varying phase ϕ. There is also an alternative way to implement the required Hamiltonian which decreases the number of laser beams to two. In this case, each laser drives a quadrupole transition such as the narrow $S_{1/2} \rightarrow D_{5/2}$ transition in ${}^{40}\mathrm{Ca}^+$ [25] and the lasers would have to be oppositely detuned from the transition by ν.

We now describe our proposal to measure a harmonic oscillator Berry phase in a vibrational mode of a trapped ion. It is assumed that the atom is cooled to the ground state of motion and in the internal state $|g\rangle$. First we prepare a coherent state $|\alpha\rangle$ of the vibrational mode. This can be done by shifting the center of the trap [13] or other methods [14,20] (our scheme is independent of the method used). Next we want to achieve the evolution

$$|g\rangle|\alpha\rangle \rightarrow |g\rangle|-\alpha\rangle \tag{14}$$

by purely geometrically. To this end we switch on the Hamiltonian H by switching on appropriate lasers and viewing the dynamics from the rotating frame with $U = \exp[-i(\nu a^\dagger a + \omega_0 \sigma_z)t]$ (we describe later what happens in the laboratory frame). The initial state can be written as

$$|\alpha\rangle|g\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} (|\Psi_n^+\rangle + |\Psi_n^-\rangle). \tag{15}$$

Next, the relative phase ϕ of the lasers is varied very slowly and cyclically while the parameter $r = \tanh^{-1} g_b/g_a$ is kept constant. If ϕ is varied slowly enough so that adiabaticity holds, the state evolves as

$$|\Psi(\phi)\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{i\gamma_n(\phi)} \\ \times [e^{i\beta_n^+(\phi)}|\Psi_n^+(\epsilon)\rangle + e^{i\beta_n^-(\phi)}|\Psi_n^-(\epsilon)\rangle], \tag{16}$$

where $\gamma_n(\phi)$ and $\beta_n^\pm(\phi)$ are geometric and dynamical phases, respectively, and $\epsilon = re^{-i\phi}$. After ϕ has completed an entire cycle (i.e., $\phi = 2\pi$), we apply a state dependent phase shift $|g\rangle \rightarrow -|g\rangle$ to the ionic state (by applying a 2π pulse [15]) and this has to be done much faster than the evolution time scale of the system. This converts $|\Psi_n^+(\epsilon)\rangle \rightarrow |\Psi_n^-(\epsilon)\rangle$ and vice versa. Now we vary the phase difference ϕ again from 2π to 4π. The resulting state at the end of this second cycle is

$$|\Psi(4\pi)\rangle = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{2}} \sum_n \frac{\alpha^n e^{i[\gamma_n(4\pi) + \beta_n^+(2\pi) + \beta_n^-(2\pi)]}}{\sqrt{n!}} \\ \times [|\Psi_n^+(\epsilon)\rangle + |\Psi_n^-(\epsilon)\rangle]. \tag{17}$$

As $\beta_n^+ = -\beta_n^-$, the dynamical phase completely cancels (for a different method of canceling the dynamical phase in a quantum evolution, see Ref. [26]). If we choose $\sinh^2 r = 1/4$, we have $\gamma_n(4\pi) = -n\pi$. Under such circumstances, the final state would be

$$|\Psi(4\pi)\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{2}} \sum_n \frac{(-\alpha^n)}{\sqrt{n!}} [|\Psi_n^+(\epsilon)\rangle + |\Psi_n^-(\epsilon)\rangle] \\ = |g\rangle|-\alpha\rangle. \tag{18}$$

So the detection of the Berry phase now amounts to distinguishing between the coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. To this end, after switching off the adiabatic evolution, the entire state is given a negative displacement of $-\alpha$. This reduces our problem to distinguishing between $|0\rangle$ (a vibrational state with no excitation) and $|-2\alpha\rangle$ [27]. To accomplish this, the ionic internal states are allowed to interact with the vibrational mode by a Jaynes-Cummings interaction [27]. In the case of no Berry phase, the probability of finding the ion in an excited state at any time t is $P_{e0} = 0$, while, in the presence of a Berry phase the same probability is given by

$$P_{e,-2\alpha} = e^{-2|\alpha|^2} \sum \frac{(-2\alpha)^{2n}}{n!} \sin^2 \frac{\Omega_{n+1}}{2} t, \tag{19}$$

where Ω_{n+1} is the Rabi frequency corresponding to an excitation number $n + 1$. Note that the above method is not a perfect discrimination (it is impossible in principle as $|0\rangle$ and $|-2\alpha\rangle$ are nonorthogonal). Incidentally, the squeezing in our proposal ($r = 0.48$) is smaller than the value $r \sim 1.5$ required for the four squeeze nonadiabatic geometric phase proposal [18]. If the original coherent state amplitude is $|\alpha| \sim 1$, we can ensure that this squeezing does not bring in a contribution from the higher order terms in the expansion of Eq. (12) (From Ref. [22] one can show that the probability of the state $|3\rangle$ in squeezing of $|1\rangle$ with $r \sim 0.48$ is already small enough to make the ratio of higher order terms to the first order term lower than $10^{-2}\eta^2$.) One alternative to using a coherent state is to start with the superposition $|0\rangle + |1\rangle$ (which should not be too difficult to prepare [28]) and convert it to the state $|0\rangle - |1\rangle$ by following an identical procedure to that described above for coherent states.

In the laboratory (nonrotating) frame, there will be an extra phase equal to $\int_c d\epsilon \langle \Psi_n^\pm | S(\epsilon)^\dagger (\nu a^\dagger a + \omega_0 \sigma_z) S(\epsilon) | \Psi_n^\pm \rangle$. For our choice of $\sinh^2 r = 1/4$ this phase turns out to be $3\nu T$, where T is the time required to complete *one* cycle of the phase ϕ. We can choose T in such a way that this phase becomes a multiple of 2π. This will keep the magnitude of the phase unaltered from that in the rotating frame.

Let us now examine the feasibility of our experiment with existing ion trap parameters. For adiabaticity, we require the time scale T of variation of the relative phase ϕ to be greater than the dynamical time scale of the problem. The dynamical time scale is given by $g_{a/b}^{-1} = (\Omega_{ij}\eta_{ij}/2)^{-1}$. We first consider experiments with the $^9\mathrm{Be}^+$ ion [12,24], where $\eta \sim 0.2$ and $\Omega/2\pi \sim 500$ kHz. The dynamical time scale, required to be smaller than T, is then about 0.33×10^{-5} s. T, on the other hand, is required to be lower than both the lifetime of the excited state (which can be up to 10 s [21]) and the motional decoherence time scale (which is about 10^{-4} s [12] for $^9\mathrm{Be}^+$ experiments) so that our assumption of neglecting all types of decoherence holds. If we set $T \sim 10^{-5}$ s (much less than both the decoherence time scales), then it is about thrice the dynamical time scale and the assumption of adiabaticity should hold (one could even try to ensure an order of magnitude difference between the dynamical time scale and by increasing the laser power a bit more than threefold). On the other hand, motional decoherence, which is an order of magnitude slower than T, can be neglected. In $^{40}\mathrm{Ca}^+$ experiments [25], the dynamical time scale is 10^{-5} s whereas the motional decoherence time scale is 10^{-3} s (internal state decoherence being 1 s). For these traps, we can choose $T \sim 10^{-4}$ s and perfectly satisfy both adiabaticity and the neglect of decoherence. We also have to set $3\nu T = m2\pi$ for the extra phase in the nonrotating frame to vanish, where m is an integer. For standard traps, $\nu \sim$ 10 MHz [20], and hence $\nu T \sim 10^4$. We can easily choose a number of such a large order to be a multiple of 2π.

It is interesting to point out that if our scheme is performed starting with the ion in the excited state, the evolution would be $|e\rangle|\alpha\rangle \rightarrow -|e\rangle|-\alpha\rangle$. Therefore, the initial internal state of the ion makes no difference to our scheme. Using a similar scheme we can also create an interesting entangled state of the ion and the vibrational mode. One initially has to prepare the ion in a superposition of $|g\rangle$ and some other state $|r\rangle$ which is completely decoupled from the evolution due to our Hamiltonian. Then the evolution proceeds as

$$(|g\rangle + |r\rangle)|\alpha\rangle \rightarrow |g\rangle|-\alpha\rangle + |r\rangle|\alpha\rangle. \qquad (20)$$

The state in the above equation is an entangled state of the ion and the vibrational mode which has attracted a lot of interest [15].

We have shown how to observe the Berry phase of a simple harmonic oscillator using a trapped ion. We have described how to reverse the amplitude of a coherent state by purely geometric means (i.e., by using only the Berry phase). An important advantage of the experiment is that it needs only a single trapped ion. This is a simple requirement in comparison to the technology that already exists such as the ability to entangle four ions in a trap [24]. Moreover, we have shown that the existing ion trap parameters are well in the range of those required for implementing our proposal and for our analysis to remain valid. We have also indicated how to use our scheme to create an entangled state of the ion and the vibrational mode. Further interesting extensions to the geometric approach to multiple modes are also possible and will be investigated in the future.

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