Ripple Formation through an Interface Instability from Moving Growth and Erosion Sources

R. Friedrich,[1] G. Radons,[2] T. Ditzinger,[1] and A. Henning[3]
[1]*Institut für Theoretische Physik III, Universität Stuttgart, D-70550 Stuttgart, Germany*
[2]*Institut für Physik, Theoretische Physik I, Technische Universität Chemnitz, D-09107 Chemnitz, Germany*
[3]*Fraunhofer IPA, Nobelstrasse 12, D-70569 Stuttgart, Germany*
(Received 19 January 2000)

The propagation of material interfaces is investigated under the action of a localized moving source which deposits or removes material. Among others the latter process applies to beam cutting techniques. We develop a Kuramoto-Sivashinsky–type model and find a new type of ripple forming mechanism. This theory offers a new explanation for the occurrence of striation patterns which often degrade the quality of cutting edges.

PACS numbers: 05.45.–a, 47.20.Ma, 79.20.Rf, 81.40.Np

The dynamics of evolving interfaces is of great interest in many disciplines ranging from physics, chemistry, materials science, and fluid mechanics to image processing and computer vision. It has been a topic of continuing research for several decades, but found renewed interest more recently in the nonlinear dynamics of pattern formation [1], fractal growth [2], and roughening phenomena [3]. In the latter context, most models for the surface dynamics $S(\mathbf{r}, t)$ in continuous space $\mathbf{r}$ and time t are defined by nonlinear partial differential equations of the form $\partial_t S(\mathbf{r}, t) = F(\nabla S(\mathbf{r}, t), \Delta S(\mathbf{r}, t), \Delta^2 S(\mathbf{r}, t), \dots) + \xi(\mathbf{r}, t)$, where ξ is a spatiotemporal noise term. The most prominent examples are the Kardar-Parisi-Zhang (KPZ) equation [4] and the Kuramoto-Sivashinsky (KS) equation [5,6] in various versions [7] without or with noise terms [8]. These models apply for instance to growth and erosion processes and describe systems with spatial translation invariance. In practical applications, however, translational invariance is broken due to the localized nature of the source which causes growth or material removal. In addition this source may be fluctuating in time or moving.

Our studies were motivated by such a problem arising in a widespread industrial application, the technique of abrasive waterjet cutting [9,10]. This machining tool is used to cut all sorts of materials ranging from titanium to ceramics and compound materials: In a focused beam of diameter $\approx$1 mm, abrasive particles (sand or garnet) are accelerated by water and air up to velocities of approximately 900 m/sec. These fast particles impinge onto the workpiece and remove material while the beam is moving with a constant feed rate of several cm/min, producing a cut with a depth of up to several centimeters (see Fig. 1). A problem with this technique is that at high feed rates, ripples and striation patterns are formed at the sidewalls of the cut which degrade the quality of the cutting edge. Similar patterns are observed with most beam cutting techniques, such as laser [11], ion, or electron beam cutting. In this paper we propose a simple model capturing essential features of the interface dynamics caused by such moving localized erosion sources.

The cutting process can be described by the dynamical evolution of the cutting surface, i.e., the interface $S(\mathbf{r}, t)$ between empty spaces, where the material has already been removed, and the solid consisting of not yet removed material. The rate of advancement of the interface depends on the eroding action of the jet. For an initially flat surface $S(\mathbf{r}, t = 0) = 0$ and for vanishing feed $\mathbf{u} = \mathbf{0}$ (drilling) this rate varies in space according to a profile function $v(\mathbf{r})$. For a jet parallel to the z axis the latter depends only on the transversal coordinates $\mathbf{r} = (x, y)$ and is typically taken as a 2D Gaussian reflecting the spatial distribution of the kinetic energy density of the impinging jet particles (energy density for laser jet cutting). With increasing time the drilling or cutting front develops structures so that dependencies of the erosion rate on the angle of impingement determined by ∇S, the curvature ΔS, and higher derivatives of the interface $S(\mathbf{r}, t)$ become relevant. Under cutting conditions the head typically moves with a

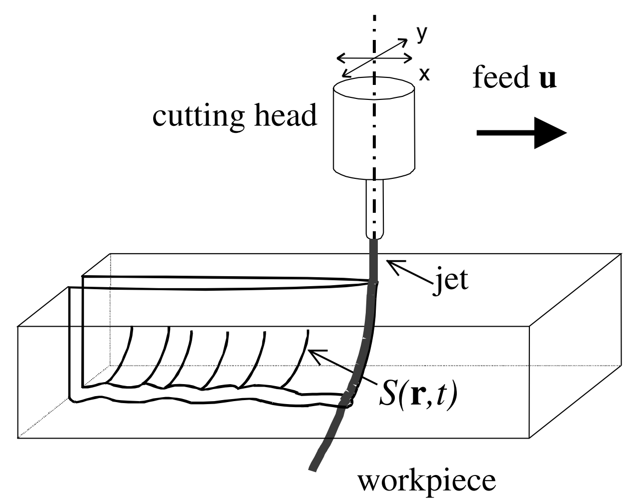

FIG. 1. The geometrical arrangement of a cutting experiment: The jet impinges perpendicularly on the workpiece while the cutting head moves with velocity $\mathbf{u}$ into the x direction. The jet causes a moving erosion front $S(\mathbf{r}, t)$ in the workpiece thus producing a cut.

 0031-9007/00/85(23)/4884(4)$15.00

constant velocity $\mathbf{u}$, i.e., the profile function has the form $v(\mathbf{r} - \mathbf{u}t)$. In a coordinate system comoving with the cutting head the temporal evolution of the interface $S(\mathbf{r}, t)$ is therefore assumed to obey the following equation in the variables $\mathbf{r} = (x, y)$ and t:

$$\partial_t S = v(\mathbf{r}) F(\nabla S, \Delta S, \Delta^2 S, \ldots) - \mathbf{u} \cdot \nabla S . \quad (1)$$

The new essential aspects lie in the localized profile function $v(\mathbf{r})$ and the appearance of the convective term $\mathbf{u} \cdot \nabla S$ from the transformation to the comoving frame. A possible noise term is neglected. An explicit form of F is obtained by assuming that the dependence of the jet-material interaction on the curvature and higher order derivatives of S is small so that one may truncate a Taylor expansion of F, i.e., $F \simeq F_0(\nabla S) + \alpha \Delta S + \beta \Delta^2 S$. A possible gradient dependence of α and β is assumed to be weak and will also be neglected. A strong dependence of the process efficiency on the impact angle, i.e., the angle between the incoming jet and the momentary surface $S(\mathbf{r}, t)$, however, is established in many of the above-mentioned applications, e.g., for erosion phenomena [12], ion sputtering [13], and laser jet cutting (see, e.g., [14]). This implies a strong gradient dependence of F_0. For erosion phenomena, as in abrasive waterjet cutting, and for brittle material the functional form $F_0 = 1/[1 + (\nabla S)^2]$ is a reasonable approximation, while for ductile workpieces the maximum wear is obtained for nonzero gradients [12]. Although we already find, depending on F_0 and $\mathbf{u}$, various interesting dynamical scenarios already for the Hamilton-Jacobi equation resulting for $\alpha = \beta = 0$ [15], no instability occurs on this level of approximation. Only the inclusion of higher order derivatives, i.e., $\alpha \neq 0$, $\beta \neq 0$, can lead to unstable behavior. The destabilizing mechanism, however, does not depend on the detailed form of F_0. For definiteness we consider cutting of brittle material with feed along the x axis, i.e., $\mathbf{u} = u\hat{\mathbf{e}}_x$, resulting in

$$\partial_t S = v(\mathbf{r}) \left(\frac{1}{1 + (\nabla S)^2} + \alpha \Delta S + \beta \Delta^2 S \right) - u S_x . \quad (2)$$

For constant v the convective term can be transformed away so that in the limit of small gradients ∇S, one recovers the well-known KS equation [5,6]. Thus our model is a generalization of the KS equation to spatially inhomogeneous processes [16].

Let us first indicate the results for the $(1 + 1)$-dimensional problem. Note in advance that for an infinitely extended homogeneous system $[v(x) = 1]$, one obtains, as a stationary solution, $S^*(x) = ax + b$, where a is the solution of the equation $ua = \frac{1}{1+a^2}$. Stability analysis shows that in this case plane wave disturbances $e^{\lambda(k)t} e^{ikx}$ decay or increase at a rate $\lambda(k) = -iku - \alpha k^2 + \beta k^4$. S^* is linearly unstable for $\beta < 0$ provided $\alpha < 0$. The wave number $k_{\max}$ of the most unstable mode is determined by $k_{\max}^2 = \alpha/(2\beta)$. Although the full nonlinear, homogeneous problem, which is not treated here, can show behavior of similar complexity as the KS equation, possibly modified by dynamical coarsening phenomena [17], the occurrence of a most unstable mode is important also for the inhomogeneous case, where the jet profile is taken as $v(x) = N \exp[-x^2/(2\sigma^2)]$ with N measuring the mean effectiveness of material removal. The region of relevant jet-material interactions of size $\approx 4\sigma$ is chosen to be of the same order as $2\pi/k_{\max}$. We may transform Eq. (2) into a dimensionless form by measuring all spatial distances in units of σ and time in units of σ/N. This transformation shows that our model is defined effectively by three independent dimensionless parameters $u' = u/N$, $\alpha' = \alpha/\sigma$, and $\beta' = \beta/\sigma^3$ [18].

We investigated the evolution equations using a semi-implicit hopscotch integration scheme (see, e.g., [19]). As expected, one finds that a positive coefficient $\alpha > 0$ always leads to a stable front. For $\alpha < 0$, however, the behavior is more complicated and depends nontrivially on the feed rate u. In the following we will consider in more detail this case of "negative surface tension," which is known to be relevant for sputter erosion [7,8,20]. For erosion of surfaces by sand a similar destabilizing mechanism involving the second spatial derivative has been proposed by arguing that the microscopic cutting action of grains is more efficient in valleys than on peaks [21]. For laser jet cutting the origin of the observed instability is less clear [14].

A first scenario for the developing cutting front is shown in Fig. 2a at four subsequent time instants. Here the impinging jet (not shown) of intensity $N = 2.5$ and width $\sigma = 2.5$ is located near $x_0 \simeq 25$, hitting the initially flat surface $S(x, t = 0) = 0$ from above. Thus the area below a curve $S(x, t)$ corresponds to material left uncut, while above $S(x, t)$ material is already removed [22]. In the comoving frame of Fig. 2 the material is constantly fed into the jet region from the right. All structures develop in the jet region, i.e., in the interval $(x_0 - 2\sigma, x_0 + 2\sigma)$, and are subsequently convected to the left. The latter follows because outside the jet the evolution equation reduces to the pure advection equation $S_t = -uS_x$ with traveling solutions $S = f(x - ut)$. The function f is determined by the processes in the interaction region. In Fig. 2a, where u was chosen as $u = -0.1$, a stable stationary front is established near the jet for large times (as in the case $\alpha > 0$). Figure 2b, in contrast, shows a very different behavior although only the feed velocity was reduced to $u = -0.06$. One finds a periodic growth of ridges in the jet region, which asymptotically leads to a left traveling wave behind the jet. In the laboratory frame, i.e., on the workpiece, this corresponds to a spatially periodic structure left over by the cutting process. The period of this structure is close to the wavelength $2\pi/k_{\max}$ of the most unstable mode of the infinite homogeneous system.

An overview of the states of the fronts for different feed rates at a late time instant is given in Fig. 3. The three upper curves lie in a stable regime, where a stationary front

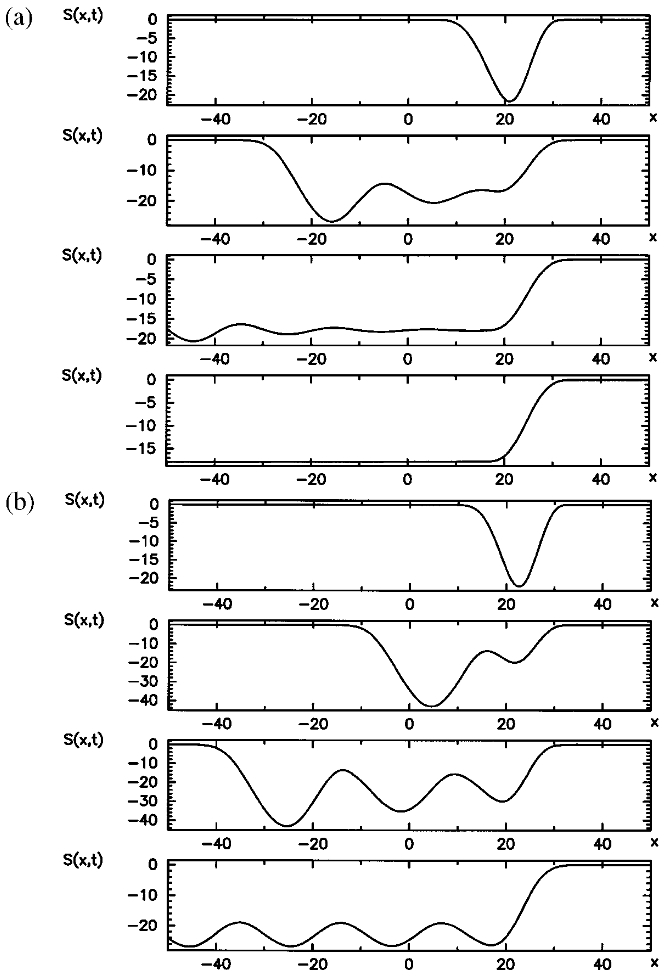

FIG. 2. Evolution of the cutting front $S(x, t)$ for time instants $t = 100$, 500, 1000, and 5000 (from top to bottom) from the initial state $S(x, t) = 0$ according to Eq. (2) for (1 + 1) dimensions ($\alpha = -1$, $\beta = -5.066$). (a) $u = -0.1$: The system evolves to a state with a stable stationary front. (b) $u = -0.06$: Asymptotically an oscillatory state is reached leaving a periodic pattern at the bottom of the workpiece.

is established. There the feed rate u influences only the height of the front, i.e., the cutting depth d_c, which varies roughly as $\sim 1/u$ as in real cutting processes [9]. Such a dependence is expected because a given location on the workpiece is exposed to the eroding action of the jet at a time period of length σ/u. From plots as in Fig. 3 (bottom) the amplitude of oscillatory solutions can also be read off. The dependence of this amplitude on the feed rate u is shown in Fig. 4. One observes that the traveling wave solution bifurcates from the stable stationary front solution. We have verified the occurrence of the bifurcation by a linear stability analysis of the stationary front, which shows that, as for a Hopf bifurcation, a conjugate complex pair of eigenvalues crosses the imaginary axis. This stabilizing influence of high feed rates is also known in the field of waterjet cutting [9]. At this point let us briefly elucidate the role of the nonlinearity $F_0 = 1/[1 + (\nabla S)^2]$ by replacing it in Eq. (2) by its linearized form $F_0 = 1$.

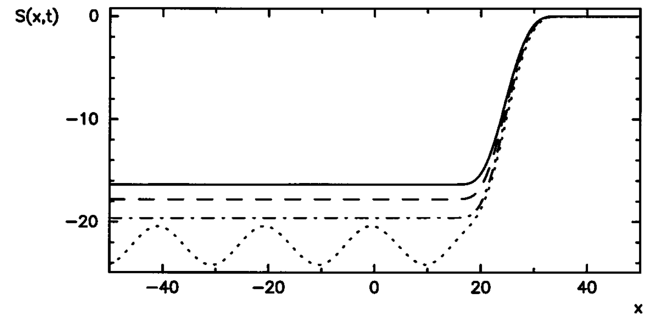

FIG. 3. The surface $S(x, t)$ at a late time $t = 20\,000$ for various feed rates $u = -0.061$, -0.081, -0.101, -0.121 (from bottom to top, other parameters as in Fig. 2). For decreasing values of u the height of the front increases, leading eventually to traveling wave behavior.

Numerical simulations of the resulting dynamics show that instabilities under the jet increase indefinitely if they exist. This clarifies the role of $F_0(\nabla S)$ in bounding the amplitude of the unstable modes. Thus both the destabilizing and the saturating mechanism act in the same spatially localized region.

Since it is not obvious that the same instability exists also in higher dimensions, we investigated the dynamics of our model also in (2 + 1) dimensions. Figure 5 shows an example of a numerical solution of the evolution equation [Eq. (2)]. One finds the same type of oscillatory instability of the stationary state for $\alpha < 0$, which eventually leads to a traveling wave state. A new aspect in (2 + 1) dimensions is that the wavelike modulation leads to spatially periodic modulations at the lateral walls of the valley, i.e., the cutting edge of the considered workpiece. The formation of such characteristic ripples is found for the waterjet cutting process [9], for laser beam cutting [11], and other beam cutting techniques.

Finally we note that more complicated, secondary bifurcation scenarios leading to quasiperiodic behavior with and without phase locking are found, if, e.g., the beam

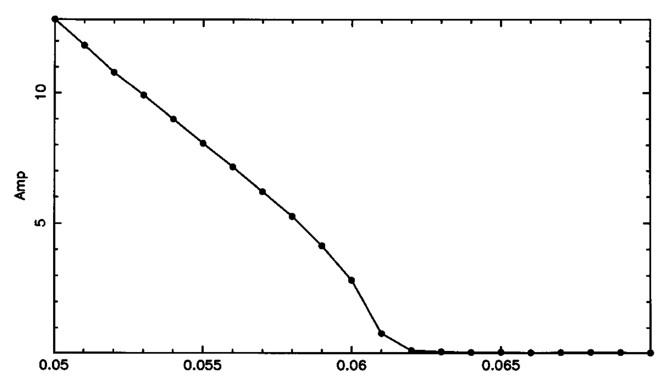

FIG. 4. Amplitude of the spatiotemporal modulation (for $t = 10^6$) as a function of the feed velocity u. The onset of an oscillatory instability of the stationary front through a Hopf-like bifurcation is found (parameters as in Figs. 2 and 3).

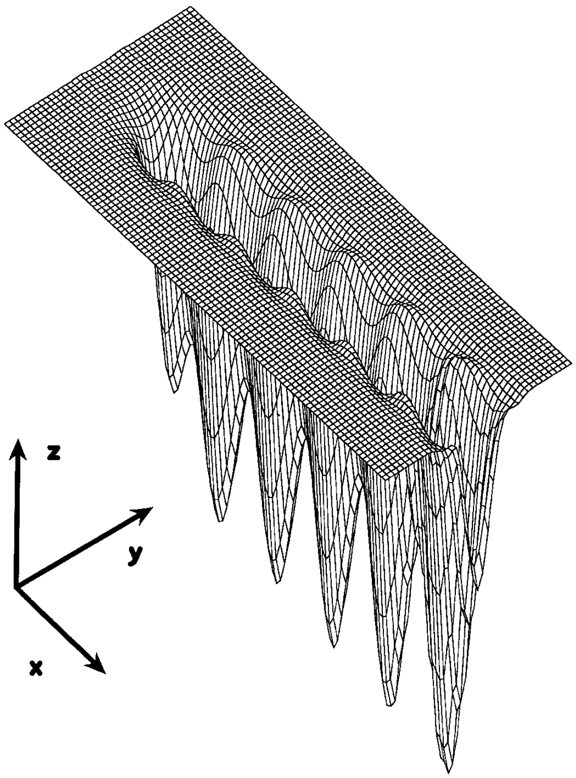

FIG. 5. Evolution of the surface $S(x, y, t)$ in (2 + 1) dimensions (parameters $\alpha = -1.8$, $\beta = -3.283$, $u = -1$, $\sigma = 3$, $N = 2.5$, and $t = 100$). The instability leads to a spatial modulation of the lateral walls (ripples on the cutting edge) and a periodic variation of the depth of the valley.

width is chosen much larger than the wavelength $2\pi/k_{\max}$. This is expected due to the complex dynamics of the related, but simpler, KS equation. In contrast to the generic Hopf-like bifurcation scenario, which we found for a wide range of parameters and which is reported in this Letter, those higher bifurcations and other interesting spatiotemporal scenarios may not have universal character. They can nevertheless be relevant for technical applications and will be treated elsewhere.

In summary we have introduced a simple model describing the evolution of interfaces under the action, e.g., of a localized moving erosion source. This model provides a new type of ripple generating mechanism, which seems to capture the essence of similar structure forming processes observed with most beam cutting techniques.

We gratefully acknowledge support from the Volkswagen-Stiftung within the priority area, "Investigation of nonlinear dynamic effects in production systems." One of us (G. R.) thanks R. Grauer and J. Krug for illuminating discussions in the early stage of this project and the Fraunhofer IPA, Stuttgart, for enabling this work.

[1] F. H. Busse and S. Müller, *Evolution of Spontaneous Structures in Dissipative Continuous Systems* (Springer-Verlag, Berlin, 1998); *Hydrodynamics and Nonlinear Instabilities,* edited by C. Godrèche and P. Manneville (Cambridge University Press, Cambridge, UK, 1998).
[2] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1992).
[3] A.-L. Barbási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, Cambridge, UK, 1995).
[4] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[5] G. Sivashinsky, Acta Astronaut. **4**, 1177 (1977).
[6] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1976).
[7] M. Rost and J. Krug, Phys. Rev. Lett. **75**, 3894 (1995).
[8] R. Cuerno and A.-L. Barabási, Phys. Rev. Lett. **74**, 4746 (1995); R. Cuerno and K. B. Lauritsen, Phys. Rev. E **52**, 4853 (1995); S. Park *et al.,* Phys. Rev. Lett. **83**, 3486 (1999).
[9] A. Momber and R. Kovacevic, *Principles of Abrasive Water Jet Machining* (Springer, Berlin, 1997).
[10] M. Hashish, Exp. Mech. **45**, 159 (1988).
[11] Y. Arata *et al.,* Trans. JWRI **8**, 15 (1979).
[12] J. G. A. Bitter, Wear **6**, 5–21 (1963); **6**, 169–190 (1963).
[13] J. L. Oechsner, Z. Phys. **261**, 37 (1973); Appl. Phys. **8**, 185 (1975).
[14] D. Schuöcker, Appl. Phys. B **40**, 9 (1986); M. Vicanek *et al.,* J. Phys. D **20**, 140 (1987); W. Schulz *et al.,* J. Phys. D **32**, 1219 (1999).
[15] For $\mathbf{u} = 0$, see, e.g., F. C. Frank, in *Growth and Perfection of Crystals,* edited by H. R. Doremus *et al.* (Wiley, New York, 1958), p. 411; J. Krug and H. Spohn, in *Solids Far from Equilibrium,* edited by C. Godrèche (Cambridge University Press, Cambridge, UK, 1992), p. 479.
[16] In contrast to other equations, such as the KS or KPZ equations [4,5,8], our model equation [Eq. (2)] is not Galilei invariant due to the inhomogeneous profile function $v(\mathbf{r})$ and the nonlinearity $F_0(\nabla S)$.
[17] C. L. Emmott and A. J. Bray, Phys. Rev. E **54**, 4568 (1996); P. Šmilauer *et al.,* Phys. Rev E **59**, R6263 (1999).
[18] Alternatively, by rescaling $x \rightarrow \lambda x$, $S \rightarrow \lambda S$ with appropriately chosen λ we may, e.g., enforce $\alpha = \beta$, leaving us with u, α, and σ as a set of dimensionless parameters.
[19] A. R. Mitchell and D. F. Griffiths, *The Finite Difference Method in Partial Differential Equations* (Wiley, Chichester, 1980).
[20] R. M. Bradley and J. M. E. Harper, J. Vac. Sci. Technol. A **6**, 2390 (1988).
[21] I. Finnie and Y. H. Habil, Wear **8**, 60 (1965).
[22] One may distinguish between interface propagation in a semi-infinite workpiece as in Figs. 2–5 ("kerfing," cut of depth $d_c \sim 1/u$), and in a plate of thickness $d < d_c$ (cutting through as in Fig. 1). The front in the latter case is obtained by removing from the full kerfing profile the part outside the plate by restricting S to values $S < d$.