Comment on "Riddling of Chaotic Sets in Periodic Windows"

The concept of a riddled basin first appeared in the literature in 1992 [1], where Alexander and co-workers noticed a surprising feature, that the basin of attraction of an attractor within an invariant manifold can be 'riddled' with a positive measure set of points belonging to the basins of other attractors. As the authors of [2] point out, this has generated much interest in the dynamics community and much research has been published on this and related phenomena [1,3,4].

Suppose that A is an invariant set under the forward dynamics of some map or flow and let $\mathcal{B}(A)$ be the basin of attraction of A. Alexander *et al.* [1] defined a basin of attraction to be *riddled* if for all $x \in \mathcal{B}(A)$ and all $\delta > 0$ we have

$$\ell(B_\delta(x) \cap \mathcal{B}(A)) > 0 \quad \text{and} \quad \ell(B_\delta(x) \cap \mathcal{B}^{\mathrm{c}}(A)) > 0, \tag{1}$$

where $B_\delta(x)$ is a ball of radius δ about the point x and $\mathcal{B}^{\mathrm{c}}(A)$ is the set of points in phase space that are not in $\mathcal{B}(A)$. Note that $\mathcal{B}(A)$ can only be riddled if the measure of $\mathcal{B}(A)$ is positive, i.e., if A is an attractor in the sense of Milnor.

The paper [2] states that the presence of a chaotic invariant set is sufficient for riddling. However, under the above definition, the basin of attraction of the chaotic invariant set cannot be riddled unless it is an attractor. Hence we must conclude that "riddling" in [2] is not riddling in the sense of [1,4].

The authors of [2] also examine F_ϵ, defined as being the probability that an initial condition chosen randomly from a line $y = \epsilon$ near the invariant manifold at $y = 0$ is asymptotic to an attractor away from $y = 0$. They numerically verify the scaling law $F_\epsilon \sim \epsilon^\gamma$, where γ is some constant. However, although riddled basins can produce this scaling, the scaling does not imply anything is riddled. For example, any map that has an unstable (repelling) fixed point p on an invariant manifold can display the same scaling if there are trajectories forming separatrices that divide the local unstable manifold of p into different regions according to whether points are asymptotic to an attractor inside or outside the invariant subspace. Moreover, one can obtain scalings in this way with any value of $\gamma > 0$, simply by adjusting the linearization at p.

In fact a basin of attraction of a linearly stable periodic orbit (or fixed point) cannot be riddled in a very general setting. Suppose that $f : M \to M$ is a continuous map, p is a linearly stable periodic orbit, and f is differentiable near p. Linear stability implies that there is an open neighborhood U of p in the basin of p. The basin of attraction of p can then be found as $\cup_{n \geq 0} f^{-n}(U)$ which is an open set, by continuity. The basin of attraction of a periodic orbit can have a highly convoluted boundary (and numerically appear to be riddled), but openness means that no riddling in the sense of [1,4] occurs. Moreover, if the only Milnor attractors in the system studied in [2] are the periodic orbit in the invariant manifold and the fixed point at infinity, neither of them can have riddled basins by the above argument. The same reasoning can be applied to show that, for the "periodic windows" considered, if the periodic orbit is the only Milnor attractor for the system in the invariant subspaces then almost all points will be in the open basin of this periodic orbit.

One could weaken the notion of riddling to say a set is riddled if there is a positive measure subset of points $x \in \mathcal{B}(A)$ satisfying (1) and use this to define a notion of a partially riddled basin. However, to say that a basin is riddled if a zero measure set of points satisfy (1) is not appropriate as this would say that a unit ball is riddled as the points on the boundary satisfy (1). We also suggest that the reason for any confusion may be explained by the fact that in a periodic window the basin of the chaotic saddle (which has zero measure) nevertheless has a dimension very close to the dimension of phase space and hence in this sense is close to being riddled.

In conclusion, we show that riddling of a basin of attraction, in the usual sense, cannot occur in the example considered by [2]. Moreover, the scaling of F_ϵ observed in that paper does not imply riddling.

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