Comment on "Scaling Laws for a System with Long-Range Interactions within Tsallis Statistics"

In their recent Letter [1], Salazar and Toral (ST) study numerically a finite Ising chain with nonintegrable interactions

$$\mathcal{H}/k_B T = -\frac{1}{2}\sum_{i\neq j}\frac{J s_i s_j}{r_{ij}^{d+\sigma}}, \qquad (s_i = \pm 1), \quad (1)$$

where $-d \le \sigma \le 0$ [2] (like ST, we discuss general dimensionality d). In particular, they explore a presumed connection between nonintegrable interactions and Tsallis' nonextensive statistics. We point out that (i) nonintegrable interactions provide no more motivation for Tsallis statistics than do integrable interactions, i.e., Gibbs statistics remain meaningful for the nonintegrable case and, in fact, provide a *complete and exact treatment*; (ii) there are undesirable features of the method which ST use to regulate the nonintegrable interactions.

ST study a system of finite length L which is simply terminated at the ends. Thus the system energy scales as $E \sim L^{d+|\sigma|}$ (or $E \sim L^d \ln L$ for $\sigma = 0$), and an intensive energy "density" is obtained from division by a "superextensive volume." We show below that the bulk free-energy density obtained from this "nonextensive thermodynamics" depends explicitly on the regulator, so all thermodynamics in this boundary-regulated model depend on boundary effects, and depend on the system shape as well for $d > 1$. A discussion of these issues is lacking in [1].

A preferable model employs periodic boundary conditions, for which a cutoff in the interaction range *must* be introduced to regulate the energy. Then the bulk nonextensive thermodynamics depend explicitly on the shape of the cutoff, but not on the boundaries or the system shape.

This homogeneous model can be solved exactly, starting from a modified (1) with interactions

$$J \to \begin{cases} R^\sigma w(r_{ij}/R) & -d \le \sigma < 0 \\ (\ln R)^{-1} w(r_{ij}/R) & \sigma = 0 \end{cases}, \quad (2)$$

where the cutoff function $w(x)$ decays at least as fast as $1/x^{|\sigma|+\varepsilon}$ ($\varepsilon > 0$) for large x, with $w(0)$ finite. This cutoff enables one to take the thermodynamic limit. The remaining problem with interactions of range R can be mapped exactly to a Kac potential for $\sigma < 0$ by identifying $\phi(x) = w(x)/x^{d+\sigma}$, so that the pair interaction is $R^{-d}\phi(r/R)$ (we present the case $\sigma = 0$ elsewhere [3]). One then takes $R \to \infty$, where the lattice becomes negligible, and recovers the *rigorous* result for the free-energy density [4]

$$f/k_B T = \text{convex envelope}\{f^0/k_B T - A m^2\}, \quad (3)$$

where f^0 is the hard-core free energy, $m = L^{-d}\sum s_i$, and

$$A = \begin{cases} \frac{\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty w(x) x^{|\sigma|-1}\,dx, & -d \le \sigma < 0 \\ \pi^{d/2} w(0)/\Gamma(d/2), & \sigma = 0 \end{cases}. \quad (4)$$

Next, consider reversing the order of limits, with $R \to \infty$ and finite L. We present only $d = 1$ for clarity. The spin s_i interacts with all periodic repeats of s_j, leading to a net pair interaction of

$$J_{ij} = R^\sigma \sum_{k=-\infty}^{\infty} \frac{w(|kL + r_{ij}|/R)}{|kL + r_{ij}|^{1+\sigma}}. \quad (5)$$

Remarkably, as $R \to \infty$ this sum converges to the constant value $(2/L)\int_0^\infty w(x) x^{|\sigma|-1} dx$, thus giving pair interactions that are independent of spatial separation: the quintessential mean-field theory. Solving this mean-field theory in the thermodynamic limit gives exactly the same free energy as before, even for general d, thus demonstrating that (3) is independent of the order of limits.

Finally, we can treat explicitly the sandwiched case $L \propto R \to \infty$ as well, by use of a hybrid of these two methods, obtaining again (3) [3]. This last limit corresponds directly to nonextensive thermodynamics, since the R^σ factor in (2) may be interpreted instead as the L-dependent temperature employed in [1]. Thus we have solved, with standard methods, the homogeneous version of nonextensive thermodynamics, while the inhomogeneous version studied in [1] would result only in *boundary-dependent* modifications to (3), due to the system-shape dependence of the cutoff function.

In summary, we find nonintegrable interactions as amenable to Gibbs statistics as integrable interactions, leaving the application of alternative methods still with the burden of motivation.

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[1] R. Salazar and R. Toral, Phys. Rev. Lett. **83**, 4233 (1999).
[2] We do not follow [1] in using "long-range" to mean nonintegrable, since "long-range interactions" already has a standard and important meaning: $0 < \sigma < 2 - \eta_{\rm sr}$.
[3] B. P. Vollmayr-Lee and E. Luijten (to be published).
[4] J. L. Lebowitz and O. Penrose, J. Math. Phys. **7**, 98 (1966).

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