Nuclear Multifragmentation in Nonextensive Statistics: Canonical Formulation

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(Received 13 March 2000; revised manuscript received 30 August 2000)

We apply the canonical quantum statistical model of nuclear multifragmentation generalized in the framework of recently proposed Tsallis nonextensive thermostatistics for the description of the nuclear multifragmentation process. The test calculation in the system with $A = 197$ nucleons shows strong modification of the "critical" behavior associated with the nuclear liquid-gas phase transition for small deviations from the conventional Boltzmann-Gibbs statistical mechanics.

PACS numbers: 25.70.Pq, 05.20.–y, 05.70.Jk, 24.60.Ky

Most of the fragmenting systems are characterized by strongly off-equilibrium processes which cease due to dissipation. The theoretical description of the fragmentation process depends on whether the equilibrium has been reached before the system starts fragmenting. If the equilibrium is attained, then the thermodynamic models using different statistical ensembles in a given fixed volume (freeze-out volume) can be applied. The ingredients specific for the considered phenomenology enter through the definition of fragment sizes and (binding) energies, fragment internal excitation properties, system size, conserved quantities in this process, etc. In nuclear physics, several models of this kind have been tried with unquestionable success in describing the transitional phenomenon in heavy ion collisions from the regime of particle evaporation at lower excitation energies to the explosion at about 5–10 MeV/nucleon of the hot source accompanied by the copious production of the intermediate mass fragments [1–3]. The situation when the fragment production has to be considered as an off-equilibrium process is described by various kinetic equations, mainly on the level of one-body distribution functions [4]. Here the statistical equilibrium is not assumed but the kinetic models in turn are plagued by the insurmountable conceptual difficulties in the calculation of asymptotic, observable features of the fragments. As an attempt to overcome at least some of these difficulties in both groups of models, in this work we extend the thermodynamic (canonical) model of the fragmentation in the framework of the recently proposed thermostatistics [5] (see [6] for recent references) to include certain off-equilibrium correlations in the system. The Tsallis' generalized statistical mechanics (TGSM), which provides the basis for generating this new model, is based on an alternative definition for the equilibrium entropy of a system whose ith microscopic state has probability $\hat{p}_i$:

$$S_q = k\,\frac{1 - \sum_i \hat{p}_i^{\,q}}{q - 1}\,, \qquad \sum_i \hat{p}_i = 1\,, \tag{1}$$

and q (entropic index) defines a particular statistics. Entropy S_q has the usual properties of positivity, equiprobability, concavity, and irreversibility and preserves the Legendre transformations structure of thermodynamics. In the limit, $q \to 1$, one obtains the usual Boltzmann-Gibbs formulation of the statistical mechanics. The main difference between the Boltzmann-Gibbs formulation and the TGSM lies in the nonadditivity of the entropy. Indeed, for two independent subsystems A, B, such that the probability of $A + B$ is factorized into $p_{A+B} = p_A p_B$, the global entropy verifies

$$S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B)\,. \tag{2}$$

TGSM provides a natural framework for the thermodynamical formalism of the anomalous diffusion and ubiquity of Levy distributions [7]. Long-range correlations in the system, as appearing in the situation of the thermalization of a hot gas penetrating in a cold gas in the presence of long-range interactions, are typical for $q > 1$ [8]. In some cases, the entropic index q in TGSM can also be related to the fluctuations of the temperature in the system [9]. A variety of the off-equilibrium situations which can be accounted for within the TGSM make it useful as a basis for the generalization of the thermodynamical fragmentation models and, in particular, in addressing the problem of the influence of these nonextensivity correlations on the signatures of "criticality" in finite systems. In the context of nuclear multifragmentation, this is usually referred to as the signatures of liquid-gas phase transition in small systems.

Our starting point is the canonical multifragmentation model [10] with the recurrence equation method [10,11] which makes the model solvable without the Monte Carlo technique and transparent to the physical assumptions and generalizations. The canonical ensemble method in TGSM was discussed in [12]. The main ingredient of the TGSM generalization of the canonical fragmentation model [10] is the expression for the fragment partition function:

$$\mathcal{Z}_q(s,t) = \sum_{\vec{p}} [1 + q_1 \beta \varepsilon_{\vec{p}}(s,t)]^{-1/q_1}, \tag{3}$$

where $q_1 \equiv q - 1$, s and t are the fragment mass number and charge number, respectively, and the fragment partition

probability equals

$$\hat{p}_{\vec{p}}(s,t) = [\mathcal{Z}_q(s,t)]^{-1}[1 + q_1\beta\varepsilon_{\vec{p}}(s,t)]^{-1/q_1}, \quad (4)$$

where $\varepsilon_{\vec{p}}(s,t) = p^2/2M + U(s,t)$, $\beta \equiv 1/T$. The internal energy U includes the fragment binding energy, the excitation energy, and the Coulomb interaction between fragments in the Wigner-Seitz approximation [2]. Changing summation into the integration in (3) one gets

$$\mathcal{Z}_q(s,t) = \frac{gV_f}{\lambda_T^3}\,\Gamma(q_1^{-1} - 3/2)\Gamma^{-1}(q_1^{-1}) \times q_1^{-3/2}[1 + q_1\beta U(s,t)]^{-\frac{1}{q_1}+3/2}, \quad (5)$$

where g is the spin degeneracy factor, V_f is the free volume, and $\lambda_T = [(2\pi)/(MT)]^{1/2}$, where M is the mass of the fragment (s,t). In the limit $q \to 1$, one recovers the familiar expression $\mathcal{Z}_1 = gV_f\lambda_T^{-3}\exp(-\beta U)$.

Given the partition function, the mean value of any quantity in TGSM is [5]

$$\langle\mathcal{O}\rangle_q = \sum_{\vec{p}} \mathcal{O}_{\vec{p}}\,\hat{p}_{\vec{p}}^{\,q}\,. \quad (6)$$

For the average energy of fragment (s,t) one obtains

$$\langle\varepsilon(s,t)\rangle_q = -\frac{\partial}{\partial\beta}\left(\frac{1 - [\mathcal{Z}_q(s,t)]^{-q_1}}{q_1}\right). \quad (7)$$

In the dilute gas approximation [13], a partition function of the whole system can be written as follows:

$$\mathcal{Q}_q(A,Z) = \sum_{\hat{n}\in\Pi_{A,Z}} \prod_{s,t} \frac{[\mathcal{Z}_q(s,t)]^{N_{\hat{n}}(s,t)}}{N_{\hat{n}}(s,t)!}\,, \quad (8)$$

where the sum runs over the ensemble $\Pi_{A,Z}$ of different partitions of A and Z of the decaying system: $\{\hat{n}\} = \{N_{\hat{n}}(1,0), N_{\hat{n}}(1,1), \ldots, N_{\hat{n}}(A,Z)\}$ and $N_{\hat{n}}(s,t)$ is the number of fragments (s,t) in the partition $\{\hat{n}\}$. In this approximation, the recurrence relation technique [10] can be applied providing the exact expression for $\mathcal{Q}_q(A,Z)$:

$$\mathcal{Q}_q(A,Z) = \frac{1}{A}\sum_{s,t<s;s-t<A-Z} s\,\mathcal{Z}_q(s,t) \times \mathcal{Q}_q(A - s, Z - t)\,. \quad (9)$$

These relations can now be conveniently used to calculate ensemble averaged characteristics. However, in order to ensure the proper normalization, it is better to work with generalized averages [12]:

$$\langle\!\langle\mathcal{O}\rangle\!\rangle_q = \langle\mathcal{O}\rangle_q/\langle 1\rangle_q\,. \quad (10)$$

These normalized mean values exhibit all convenient properties of the original mean values (6). Moreover, when the normalized mean values (10) are used, the TGSM can be reformulated in terms of ordinary linear mean values calculated for the renormalized entropic index $q^* = 1 + (q-1)/q$. In particular, the total average energy of the system becomes

$$\mathcal{E}_q = \sum_{s,t}\langle N(s,t)\rangle_{q^*AZ}\langle\varepsilon(s,t)\rangle_{q^*}\,, \quad (11)$$

where $\langle\varepsilon(s,t)\rangle_{q^*}$ is given in (7) and

$$\langle N(s,t)\rangle_{qAZ} = \mathcal{Z}_q(s,t)\,\frac{\mathcal{Q}_q(A - s, Z - t)}{\mathcal{Q}_q(A,Z)}\,. \quad (12)$$

Analogously, the heat capacity ($= \partial\mathcal{E}_q/\partial T|_V$) is

$$C_V = \beta^2\Bigg\{\sum_{s,t}\sum_{s',t'}\langle\Delta(st;s't')\rangle_{q^*}\langle\varepsilon(s,t)\rangle_{q^*} \times\langle\varepsilon(s',t')\rangle_{q^*} + \sum_{s,t}\langle N(s,t)\rangle_{q^*AZ} \times[\langle\varepsilon^2(s,t)\rangle_{q^*} - \langle\varepsilon(s,t)\rangle_{q^*}^2]\Bigg\}, \quad (13)$$

where

$$\langle\Delta(st;s't')\rangle_q \equiv \langle N(s,t)N(s',t')\rangle_{qAZ} - \langle N(s,t)\rangle_{qAZ}\langle N(s',t')\rangle_{qAZ}$$

and

$$\langle N(s,t)N(s',t')\rangle_{qAZ} = \mathcal{Z}_q(s,t)\mathcal{Z}_q(s',t')\,\frac{\mathcal{Q}_q(A - s - s', Z - t - t')}{\mathcal{Q}_q(A,Z)} + \delta_{ss'}\delta_{tt'}\mathcal{Z}_q(s,t)\,\frac{\mathcal{Q}_q(A - s, Z - t)}{\mathcal{Q}_q(A,Z)}\,. \quad (14)$$

Figure 1 shows the caloric curve for different values of the entropic index $q \geq 1$ in the above described canonical multifragmentation model. The calculations are done for the system with $Z_0 = 79$ protons and $N_0 = 118$ neutrons ($A_0 = 197$). The free volume is $V_f = 3A_0/\rho_0 \equiv A_0/\rho_f$, where $\rho_0 = 0.168$ fm^{-3}. The excitation energy is $E^* = \mathcal{E}_q(T,\rho_f) - \mathcal{E}_q(T = 0,\rho_0)$. Curves $T(E^*/A_0)$ for different q are very similar outside of the "critical zone" of excitation energies: $E^*/A_0 \in (2.5\text{–}10)$ MeV. On the other hand, a strong sensitivity to even tiny changes of q can be seen inside the critical zone. The same data of the ALADIN Collaboration for the reaction ^{197}Au + ^{197}Au at $E_{\rm lab}/A = 600$ MeV [14] have been analyzed independently by Műller *et al.* [15] (filled points) and Majka *et al.* [16] (open points) using different prescriptions for the sequential decay correction and for the determination of the excitation energy. This causes the differences mainly at higher temperatures and excitation energies. In this experiment, mass/charge of the fragmenting system varies with the excitation energy due to the participant-spectator geometry of the reaction. The data exhibit a plateau which is seen in our model for $q \gtrsim 1.0005$.

This fragility of equilibrium ($q = 1$) "critical behavior" to the small changes of the entropic index can be seen even better in Fig. 2 which shows the specific heat vs the temperature. The increase of the entropic index q is associated with both a significant sharpening of the peak in C_V and an increase of the "critical" temperature T_C. Since this increase of T_C is accompanied by only a small change of the

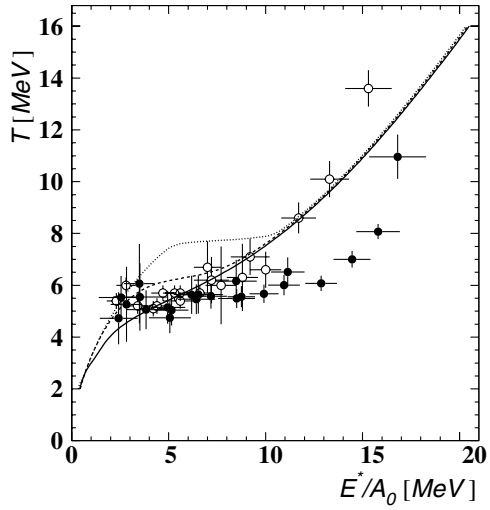

FIG. 1. The "caloric curve" for the system with $A_0 = 197$ nucleons is plotted for various values of the entropic index q: 1 (the solid line), 1.0005 (the dashed line), and 1.001 (the dotted line). For the description of the ALADIN data (filled [15] and empty [16] points), see the text.

total excitation energy (see Fig. 1), therefore the kinetic part in the total energy increases with q. In other words, the multifragmentation in statistical systems with $q > 1$ takes place in the hotter environment than in the limiting equilibrium case $q = 1$.

It is an open question whether the correlations for $q \neq 1$ change the nature of the equilibrium "phase transition." Whereas the liquid-gas phase transition is characterized by properties of the largest cluster [17], the shattering phase transition in off-equilibrium systems is characterized by the multiplicity of fragments [18]. Figure 3 shows the average multiplicity dependence of the normalized second factorial cumulant moment of the multiplicity distribution: $\gamma_2 = (\langle m(m-1)\rangle - \langle m\rangle^2)/\langle m\rangle^2$. γ_2 is a measure of the fragment-fragment correlations and equals 0 for the Poisson distribution. One can see a strong buildup of multiplicity fluctuations with increasing q in the "critical region." This enhancement of γ_2 is associated with the strong peak in C_V as seen in Fig. 2. For $q = 1.001$, the maximum of γ_2 is comparable with those found in the 2D and 3D percolation systems of comparable size [19].

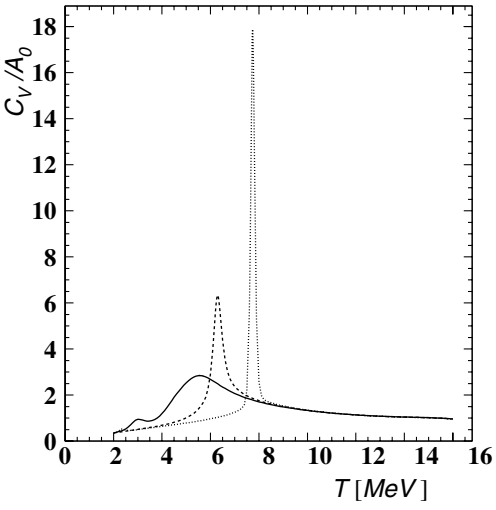

FIG. 2. The specific heat is plotted for various values of the entropic indices q. For the description of lines, see the caption of Fig. 1.

The fragment-size distributions dN/dA at $T \sim T_C$ for different q values are shown in Fig. 4. One can see the significant evolution of dN/dA with an increasing entropic index which, together with the evolution of multiplicity distributions (see Fig. 3), illustrate the change of mechanism of the multifragmentation. At $T = T_C$, one finds approximately a powerlike fragment-size distribution for $q = 1$ and the persistence of the heaviest residue for $q > 1$. The critical zone for $q > 1$ is associated with the exponential fragment-size distribution, like the inside of the spinodal region of the liquid-gas phase diagram or in the off-equilibrium critical binary fragmentation process with the Gaussian dissipation [20]. For $T/T_C > 1$ and $q > 1$, the heavy residue explodes into the large number of light fragments and the fragment-size distribution remains exponential.

In conclusion, we have developed the generalization of the canonical multifragmentation model in TGSM. This new model provides an alternative way of thinking about intuitively expected deviations from the thermodynamical equilibrium due to nonextensive correlations in the multifragmentation process. We see the main advantage of the proposed approach in the preservation of both the mathematical structure of thermodynamics and the correct description of fragments as in the standard thermodynamical models. Variability of different signals of equilibrium phase transition with respect to small deviations from

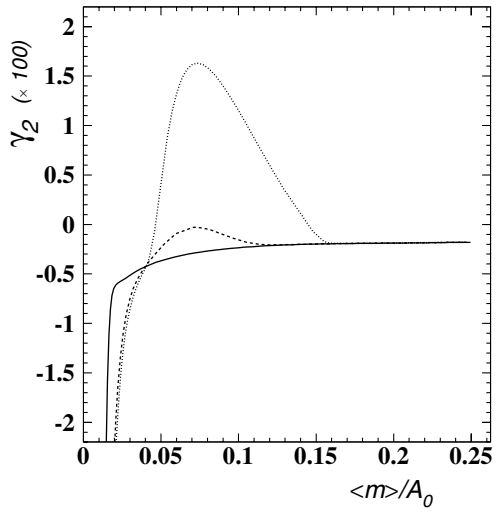

FIG. 3. The normalized second factorial cumulant moment of the fragment multiplicity distribution is plotted as a function of the average multiplicity for various entropic indices q. For other information, see the caption of Fig. 1.

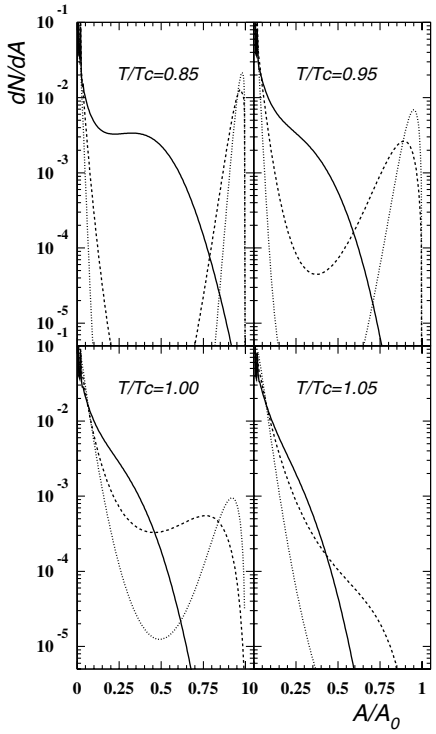

FIG. 4. The fragment-size (mass) distribution is plotted for different entropic indices and temperatures normalized by T_C. For the description of lines, see the caption of Fig. 1.

$q = 1$ demonstrates that the characterization of nuclear phase transition in terms of finite-size scaling analysis may be hazardous. All considered characteristics change qualitatively with q in the critical zone and one cannot exclude that even the order of the phase transition changes. It should be stressed that the small nonextensive corrections considered in this work would be impossible to detect in the particle/fragment energy spectra which for very small deviations from $q = 1$ would remain exponential as in the ordinary equilibrium thermodynamics. The new flexible family of fragmentation models obeying q statistics provides a powerful tool in analyzing experimental data and in characterizing possible deviations from an idealized equilibrium phase-transition picture in nuclear multifragmentation in terms of the entropic index.

We thank the ALADIN Collaboration for providing their data prior to publication. We thank also W. Trautmann for helpful comments. The work was supported by the IN2P3-JINR Agreement No. 00-49.

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