Comment on "Regular Black Hole in General Relativity Coupled to Nonlinear Electrodynamics"

In their Letter [1] Ayón-Beato and García suggested an example of a static, spherically symmetric solution of general relativity coupled to nonlinear electrodynamics (NED) with the Lagrangian $L(f)$, where $f = -F_{\mu\nu}F^{\mu\nu}$. The metric is globally regular and, for certain electric charge to mass ratios, describes black holes with a global structure similar to Reissner-Nordström and a proper spatial asymptotic but with a regular center instead of a singularity. Two other examples of solutions having quite similar properties were also given [2].

Meanwhile, it was proved long ago [3] that precisely this theory cannot lead to solutions with a regular center for whatever function $L(f)$ is chosen; it was required only that L should have a Maxwell asymptotic at small f: $L \sim f$, $L_f = dL/df \to$ const. Before explaining this seeming contradiction, I will briefly outline a proof.

Since the Ricci tensor for a static, spherically symmetric metric is diagonal, the invariant $R_{\mu\nu}R^{\mu\nu} \equiv R^\nu_\mu R^\mu_\nu$ is a sum of squares; hence each R^μ_μ (no summing) is finite at a regular point, and the same is true for the energy-momentum tensor. For NED with $L = L(f)$ and an electric radial field this leads to $|fL_f| < \infty$. On the other hand, the electromagnetic field equations imply $fL_f^2 = 2q^2/r^4 \to \infty$ as $r \to 0$ (q is a charge). If the center $r = 0$ is regular, the two conditions for f and L_f combine to give $f \to 0$, $L_f \to \infty$, i.e., a non-Maxwell behavior at small f, contrary to the assumption. This theorem admits an extension to dyonic configurations [4].

Now, the existence of the above "regular" solutions can be explained as follows: there are different functions $L(f)$ at large and small r, and the second one is non-Maxwellian as $f \to 0$. An inspection shows that this is indeed the case.

The solutions of [1,2] were found using an alternative form of NED, obtained from the original one by a Legendre transformation: one introduces the tensor $P_{\mu\nu} = L_f F_{\mu\nu}$ with its invariant $p = -P_{\mu\nu}P^{\mu\nu}$ and considers the Hamiltonian-like quantity $H = 2fL_f - L$ as a function of p; the whole theory is reformulated in terms of p and is specified by $H(p)$. One has then

$$L = 2pH_p - H, \qquad L_f H_p = 1, \qquad f = pH_p^2, \quad (1)$$

with $H_p = dH/dp$. It is a good framework for obtaining explicit solutions (quadratures already exist in a general form). Indeed, $P_{\mu\nu}$ "ignores" the nonlinearity, giving $p = 2q^2/r^4$, while the metric has the form $ds^2 = Adt^2 - A^{-1}dr^2 - r^2 d\Omega^2$, where

$$A = 1 - \frac{2M(r)}{r}, \qquad M(r) = \frac{1}{4}\int H(p) r^2\, dr\,. \quad (2)$$

It remains to choose $H(p)$ and to substitute $p = 2q^2/r^4$, so quite numerous examples may be built. A regular center exists if and only if H has a finite limit as $p \to \infty$, and a unique mass value that provides regularity for given q is then found by integrating from 0 to ∞ in (2).

For any regular solution, however, f vanishes at $r = 0$ and $r \to \infty$ and inevitably has at least one maximum, $f = f^* > 0$, at some $p = p^* > 0$. But the f and p frameworks are equivalent only in ranges of p where $f(p)$ is monotonic. One can show [4] that at an extremum of $f(p)$ where $f > 0$ the derivative L_f has the same finite limit as $p \to p^* + 0$ and $p \to p^* - 0$, while L_{ff} tends to infinities of opposite signs. The function $L(f)$ suffers branching, and its graph forms a cusp at $f = f^*$; different functions $L(f)$ correspond to $p > p^*$ and $p < p^*$.

Another kind of branching occurs at extrema of $H(p)$, if any: there $f(p)$ behaves generically as $(p - p^*)^2$ while $L_f \to \infty$, and a graph of $L(f)$ smoothly touches the vertical axis $f = 0$.

The number of Lagrangians on the way from infinity to the center equals the number of monotonicity ranges of $f(p)$. At junctions the electromagnetic field shows a singular behavior, well revealed using the effective metric [5] in which NED photons move along null geodesics. This metric is generically singular at extrema of $f(p)$, and the effective potential for geodesics exhibits poles, as was shown in [5] for the special solution of [1] and can be shown for a generic theory [4].

Thus the solutions of [1,2] and the like, seeming to be well behaved in the p framework, cannot correspond to a fixed Lagrangian $L(f)$ and show undesired features at certain p. The p framework is secondary; its equations are found by substitution from the f framework where the Lagrangian dynamics is formulated.

The findings of [1,2] reveal an underwater stone existing in NED and probably in other similar theories obtained in the field limits of string theory, M theory, etc.

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