Novel Soliton Solutions of the Nonlinear Schrödinger Equation Model

Vladimir N. Serkin[1] and Akira Hasegawa[2]
[1]*Benemerita Universidad Autonoma de Puebla, Instituto de Ciencias, Apdo Postal 502, 72001 Puebla, Puebla, Mexico*
[2]*Soliton Communications, #403, 19-1 Awataguchi Sanjobocho, Higashiyama-ku, Kyoto, 605-0035, Japan*
(Received 26 April 2000)

The methodology developed provides for a systematic way to find an infinite number of the novel stable bright and dark "soliton islands" in a "sea of solitary waves" of the nonlinear Schrödinger equation model with varying dispersion, nonlinearity, and gain or absorption. It is shown that solitons exist only under certain conditions and the parameter functions describing dispersion, nonlinearity, and gain or absorption inhomogeneities cannot be chosen independently. Fundamental soliton management regimes are discovered.

PACS numbers: 05.45.Yv, 42.65.Re, 42.65.Tg, 42.81.Dp

The nonlinear Schrödinger equation model (NLSE) is one of the most important and "universal" nonlinear models of modern science. NLSE appears in many branches of physics and applied mathematics, including nonlinear quantum field theory, condensed matter and plasma physics, nonlinear optics and quantum electronics, fluid mechanics, theory of turbulence and phase transitions, biophysics, and star formation. The current state of the art in this very active field is reviewed, for instance, in [1,2]. The best known solutions of the NLSE are those for solitary waves, or solitons. Characteristic properties of solitons include a localized wave form that is retained upon interaction with other solitons, giving them a "particlelike" quality. The theory of NLSE solitons was developed for the first time in 1971 by Zakharov and Shabad [3]. Over the years, there have been many significant contributions to the development of the NLSE solitons theory (see, for example, [1–9] and references therein). After predictions of the possibility of the existence [10] and experimental discovery by Mollenauer, Stolen, and Gordon [11], today, NLSE optical solitons are regarded as the natural data bits and as an important alternative for the next generation of ultrahigh speed optical telecommunication systems [12–17].

In this Letter, we predict a new type of nonlinear Schrödinger solitary waves. We find that there exists an effective mathematical algorithm to discover and investigate an infinite number of novel solitary wave solutions for the NLSE model with varying dispersion, nonlinearity, and gain or absorption.

The problem of soliton management described by the NLSE model with varying coefficients,

$$i\,\frac{\partial\Psi}{\partial Z} \pm \frac{1}{2}\,D(Z)\,\frac{\partial^2\Psi}{\partial T^2} + R(Z)\,|\Psi|^2\Psi = i\Gamma(Z)\Psi\,, \quad (1)$$

is a new and important one (see, for example, the review of optical solitons dispersion management principles and research as it currently stands in [18–21], and references therein). NLSE (1) is written here in standard soliton units, as they are commonly known. There is an assumption that the perturbations to the dispersion parameter $D(Z)$, nonlinearity $R(Z)$, and to the amplification or absorption coefficient $\Gamma(Z)$ *are not limited* to the regime where they are smooth and small. If we compare Eq. (1) with the quantum mechanical Schrödinger equation and replace Z by t and T by x, we recognize that varying coefficients in Eq. (1) represent the equivalent time-dependent external potentials for the quasiparticle wave function $\Psi(x,t)$.

Theorem 1.—Consider the NLSE model (1) with varying dispersion, nonlinearity, and gain or absorption. Suppose that the Wronskian $W\{R,D\}$ of the functions $R(Z)$ and $D(Z)$ is nonvanishing, the two functions $R(Z)$ and $D(Z)$ are thus linearly independent. There is then an infinite number of solitary wave solutions for Eq. (1) written in the following form:

$$\Psi = D^{1/2}R^{-1/2}PQ(S)\exp\left[i\,\frac{P}{2}\,T^2 + i\int_0^Z K(\zeta)\,d\zeta\right], \quad (2)$$

where the real function $Q(S)$ describes a canonical form of bright [sgn $= +1$, $Q(P(Z)T) = \eta\,\mathrm{sech}(\eta P(Z)T)$] or dark [sgn $= -1$, $Q(P(Z)T) = \eta\tanh(\eta P(Z)T)$] NLSE solitons, and the real functions $D(Z)$, $R(Z)$, $\Gamma(Z)$, and $P(Z)$ satisfy the equation system

$$P_Z + P^2 D = 0; \qquad W - PRD^2 = 2\Gamma RD\,. \quad (3)$$

Theorem 2.—Consider the NLSE model (1) with varying dispersion, nonlinearity, and gain or absorption. Suppose that the Wronskian $W\{R,D\}$ is vanishing, the two functions $R(Z)$ and $D(Z)$ are thus linearly dependent. There are then an infinite number conserving the pulse area solitary wave solutions for Eq. (1):

$$\Psi = CPQ(S)\exp\left[i\,\frac{P}{2}\,T^2 + i\int_0^Z K(\zeta)\,d\zeta\right], \quad (4)$$

where the real function $Q(S)$ describes a canonical form of bright or dark NLSE solitons, and the real functions $P(Z)$, $D(Z)$, $R(Z)$, and $\Gamma(Z)$ satisfy the equation system

$$P_Z - 2\Gamma P = 0; \qquad D = C^2R = -2\Gamma P^{-1}. \quad (5)$$

To prove Theorems 1 and 2 we first construct a stationary localized solution for the NLSE model (1). Substituting

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ansatz (2) into Eq. (1) we obtain

$$\pm\frac{1}{2}\frac{\partial^2 Q}{\partial S^2} + Q^3 + \left(E - \frac{S^2}{2}\Omega^2\right)Q = 0, \tag{6}$$

$$\frac{\partial P}{\partial Z}Q + P\frac{\partial Q}{\partial S}\frac{\partial S}{\partial Z} + \frac{1}{2}\frac{W\{R,D\}}{DR}PQ = \Gamma PQ - \frac{1}{2}DP^2Q - DP^2T\frac{\partial Q}{\partial S}\frac{\partial S}{\partial T}. \tag{7}$$

Here we use the notations for the Wronskian $W\{R,D\} = RD_Z - DR_Z$ and for $S(Z,T) = P(Z)T$. Equation (6) represents the nonlinear wave equation for the Schrödinger harmonic oscillator, where we use the notations for the energy $E(Z) = -K/P^2/D$ and for the frequency $\Omega^2(Z) = D^{-1}P^{-2}(P^{-2}P_Z + D)$. Equation (6) was solved numerically for the first time in [22] and gave rise to a concept of quasisolitons [18].

Now we make the important assumption about the solution of the equation system (6) and (7). Let us consider the complete nonlinear regime when Eq. (6) represents the ideal NLSE. Then from (6) and (7) it follows equation system (3). We note that the main varying parameters $\Gamma(Z)$, $D(Z)$, and $R(Z)$ are characterized by the value of the Wronskian $W\{R,D\}$. It is then straightforward to verify systems of Eqs. (3) and (5) after using the conditions $W \neq 0$ in case (3) or $W = 0$ in case (5). Theorem 1 as well as Theorem 2 have now been proved.

One can easily construct the more general solution for the NLSE model (1) in the case of arbitrary soliton amplitude and velocity by applying the Galilei transformation of Eqs. (2) and (4) and using the equation for the "soliton center" $\Delta(Z)$ given by $\Delta_Z = -VD(Z)$, where V is a soliton group velocity (in the case of spatial soliton $V = \tan\theta$, and θ is the angle of propagation in the X-Z plane).

Now we turn our attention to finding solutions for specified soliton management conditions.

Case 1: soliton dispersion management.—Suppose that dispersion management function $D(Z) = \Phi(Z)$ (we call it control function here) is a known arbitrary analytical function, and nonlinearity $R(Z) = \text{const}$. The function $\Phi(Z)$ is required only to be a once-differentiable and once-integrable, but otherwise arbitrary, function; there are no restrictions. There are then an infinite number of solutions for Eq. (1) of the form of bright and dark dispersion managed (DM) solitons represented by Eq. (2), where the main functions $P(Z)$ and $\Gamma(Z)$ are given by

$$P = -\left[C - \int \Phi(Z)\,dZ\right]^{-1}; \qquad 2\Gamma = \Phi^{-1}\Phi_Z - P\Phi, \tag{8}$$

and C is the constant of integration (both positive and negative).

Case 2: soliton amplification management and the problem of optimal soliton amplification.—Suppose that the gain (or loss) coefficient is determined by the known control function $\Gamma(Z) = \Lambda(Z)$, where the control function $\Lambda(Z)$ is required only to be once integrable. There are then an infinite number of solutions for Eq. (1) of the form of bright and dark solitons represented by Eq. (2), where the main functions $D(Z)$ and $P(Z)$ are given by quadratures

$$|P(Z)|\,|D(Z)| = \exp\left[\int 2\Lambda(Z)\,dZ + C_1\right], \tag{9}$$

$$\ln|D| = \int [2\Lambda(Z) \pm |P(Z)|\,|D(Z)|]\,dZ + C_2. \tag{10}$$

Here integration constants $C_{1,2}$ are determined by initial conditions.

Case 3: soliton pulse width management and the problem of optimal soliton pulse compression.—Suppose that soliton pulse width is determined by the known control function $P(Z) = \Theta(Z)$, where the real function $\Theta(Z)$ is required only to be a twice-differentiable, but otherwise arbitrary, function, there are no restrictions. There are then an infinite number of solutions for Eq. (1) of the form of bright and dark solitons represented by Eq. (2), where the main coefficients of the NLSE model $D(Z)$ and $\Gamma(Z)$ are given by

$$D(Z) = -\Theta^{-2}\Theta_Z; \qquad 2\Gamma(Z) = \Theta_Z^{-1}(\Theta^{-1}\Theta_Z)_Z. \tag{11}$$

Case 4: combined nonlinear and dispersion soliton management regime.—To suppose that the Wronskian $W\{R,D\}$ is vanishing means that the nonlinearity $R(Z)$ and dispersion $D(Z)$ are linearly dependent functions. Assume also that dispersion management function $D(Z)$ is determined by the known control function $D(Z) = \Xi(Z)$, where the function $\Xi(Z)$ is required only to be once integrable. There are then an infinite number of solutions for Eq. (1) of the form of bright and dark conserving pulse area solitons represented by Eq. (4), where the main functions $D(Z)$, $P(Z)$, $R(Z)$, and $\Gamma(Z)$ are given by quadratures

$$P(Z) = -\left[C - \int \Xi(Z)\,dZ\right]^{-1}, \tag{12}$$

$$2\Gamma(Z) = -\Xi(Z)P(Z); \qquad D(Z) = C^2R(Z). \tag{13}$$

Notice that by applying Theorem 1 [Eqs. (2) and (3)] and Theorem 2 [Eqs. (4) and (5)] we can find a fundamental set of different soliton management regimes—for example, soliton dispersion management, soliton energy and intensity control, optimal soliton amplification, and compression regimes.

The interested reader can take different arbitrary parameter functions $D(Z)$, $R(Z)$, or $\Gamma(Z)$ to find the novel "soliton islands" in a "sea of solitary waves" for the NLSE model (1) by using the algorithm developed in this paper. Notice that soliton solutions exist only under certain conditions and the parameter functions $D(Z)$, $R(Z)$, and $\Gamma(Z)$

cannot be chosen independently; they satisfy equation systems (3) and (5).

Examples.—Let us consider some examples. The fundamental set of dispersion managed solutions can be expressed in trigonometric and hyperbolic functions. Assume that the dispersion coefficient of the NLSE model (1) is a periodically varying control function:

$$D(Z) = \Phi(Z) = 1 + \delta \sin^m \kappa Z . \tag{14}$$

Then an infinite number of the DM-soliton solutions are given by Eqs. (2) and (8). Integration in (8) is elementary for any value of the parameter m, and in the simplest case ($m = 1$) is given by

$$P(Z)^{-1} = -[C - Z + \delta \cos(\kappa Z)/\kappa], \tag{15}$$

$$2\Gamma(Z) = -\Phi P + \Phi^{-1}\delta\kappa \cos\kappa Z . \tag{16}$$

Let us consider periodical soliton dispersion management regimes in the case of Theorem 2. The main feature of soliton solutions given by Theorem 2 consists of the fact that the soliton pulse area is conserved. Assume the dispersion inhomogeneity $D(Z)$ to be a potential barrier, for instance, of the cos or sin functional form. Then the combined nonlinear and dispersion management regime is given by

$$D = \cos Z; \qquad P = -(C - \sin Z)^{-1}; \qquad 2\Gamma = -PD , \tag{17}$$

where arbitrary constant $|C| > 1$.

Let us consider the soliton pulse width management regimes. One of the simplest periodical soliton solutions is given by

$$P(Z) = \Theta(Z) = -(1 + \delta \sin^2 Z); \qquad 2\Gamma = -PD , \tag{18}$$

$$D(Z) = \delta(1 + \delta \sin^2 Z)^{-2} \sin 2Z . \tag{19}$$

The main features of analytical solutions predicted (Theorems 1 and 2) have been investigated by using direct computer simulations. Their solitonlike features have been proved in our computer simulations with the accuracy as high as 10^{-9}. The time-space evolution of bright and dark DM solitons for the case represented by Theorem 1 [Eqs. (2) and (14)–(16)] is shown in Fig. 1. The time-space dynamics of the propagation and interaction of DM solitons for the case of Theorem 2 [Eqs. (4), (18), and (19)] is shown in Fig. 2. An important feature of the solitary waves solutions given by Eqs. (2)–(5) consists of the

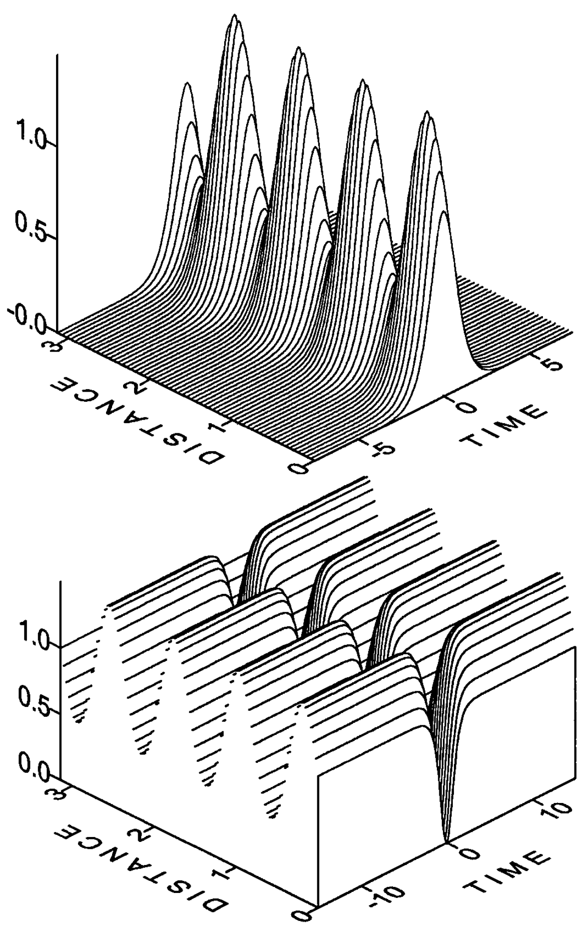

FIG. 1. Evolution of the dispersion managed bright and dark solitary waves [Eqs. (2) and (14)–(16)] as a function of the propagation distance. Initial conditions: $m = 1$; $\kappa = 8$; $\delta = 0.5$; and $C = 10^4$.

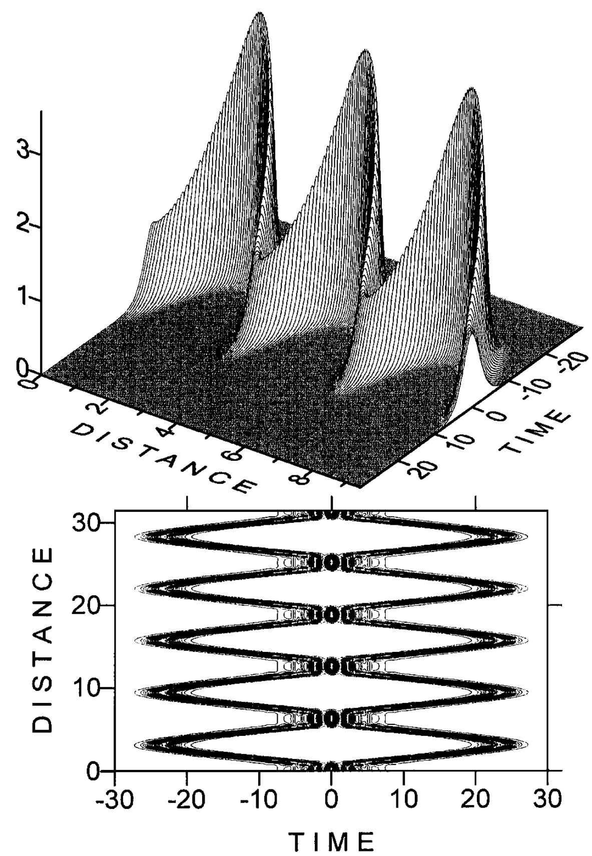

FIG. 2. Evolution of the DM bright solitary waves [Eqs. (4), (18), and (19)] as a function of the propagation distance (DM-soliton "snake" effect) and nonlinear trapping of two DM-soliton pulses in the periodically dispersion managed structure [Eq. (19)]. Initial conditions: $\Gamma(Z = 0) = 0.5$; $\delta = 0.9$; and $V_{1,2}(Z = 0) = \pm 10$.

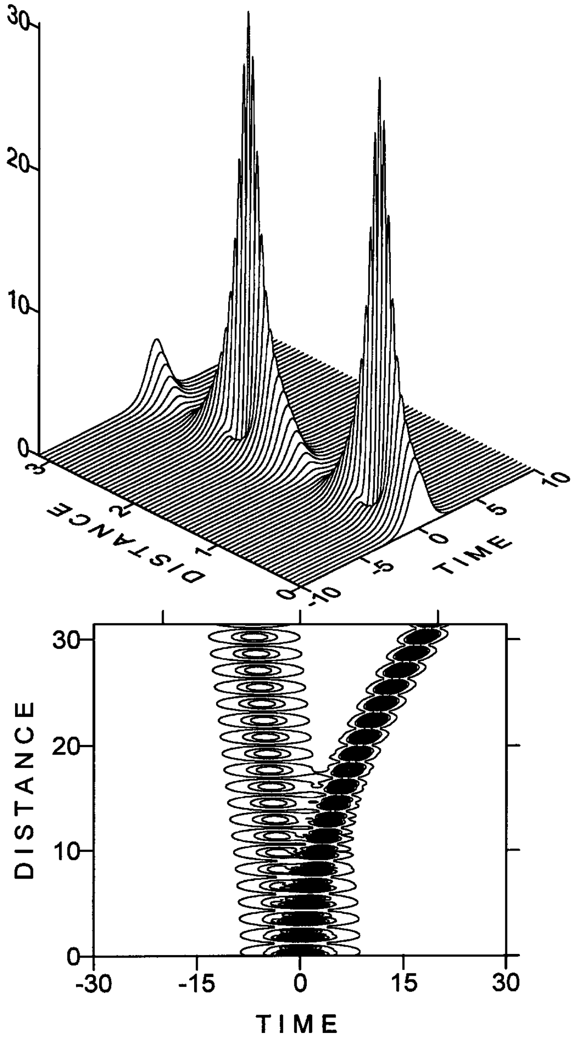

FIG. 3. Two DM solitons [Eqs. (2) and (14)–(16)] bound state formation and decay in the presence of Raman self-scattering effect. Initial conditions: $\kappa = 4$; $C = 10^4$; and $\delta = 0.9$.

elastic character of their interaction. We also have investigated the nonlinear dynamics of high-order solitons generation in the frame of the NLSE model (1). Figure 3 shows periodical time-space evolution of the bound state of two DM solitons [Eqs. (2) and (14)–(16)] and represents the decay of this bound state produced by a self-induced Raman soliton scattering effect which has been considered within the framework of the oscillator model [23]. This remarkable fact also emphasizes the full soliton features of solutions discussed. They not only interact elastically but they can form bound states, and these bound states split under perturbations.

In summary, the methodology developed (Theorems 1 and 2) provides for a systematic way to discover and investigate an infinite number of the novel solitary waves for the NLSE model with varying dispersion, nonlinearity, and gain or absorption. The surprising aspect is that analytical solutions are obtained here in quadratures. Their pure solitonlike features are confirmed by accurate direct computer simulations. The results obtained in this Letter are of general physics interest and should be readily experimentally verified. The finding of a new mathematical algorithm to discover solitary wave solutions in nonlinear dispersive systems with spatial parameter variations is important to the field, and might have significant impact on future research.

Finally, the authors would like to express special gratitude to Professor V. E. Zakharov for reading and commenting on the entire manuscript and for fruitful suggestions. Special thanks are due to Professor T. H. Tieman for carefully checking the manuscript.

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