

How to Measure the Bogoliubov Quasiparticle Amplitudes in a Trapped Condensate

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We propose an experiment, based on two consecutive Bragg pulses, to measure the momentum distribution of quasiparticle excitations in a trapped Bose gas at low temperature. With the first pulse one generates a bunch of excitations carrying momentum q , whose Doppler line is measured by the second pulse. We show that this experiment can provide direct access to the amplitudes u_q and v_q characterizing the Bogoliubov transformations from particles to quasiparticles. We simulate the behavior of the nonuniform gas by numerically solving the time dependent Gross-Pitaevskii equation.

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More than 50 years ago Bogoliubov [1] developed the microscopic theory of weakly interacting Bose gases. A crucial step of the theory is given by the so called Bogoliubov transformations

$$b_{\mathbf{q}} = u_q a_{\mathbf{q}} + v_q a_{-\mathbf{q}}^\dagger, \quad (1)$$

$$b_{\mathbf{q}}^\dagger = u_q a_{\mathbf{q}}^\dagger + v_q a_{-\mathbf{q}}, \quad (2)$$

which transform particle creation a and annihilation a^\dagger operators into the corresponding quasiparticle operators b and b^\dagger . The real coefficients u_q and v_q are known as quasiparticle amplitudes. The Bogoliubov transformations are the combined effect of gauge symmetry breaking and of the interactions which are responsible for the mixing between the particle creation and annihilation operators. In virtue of transformations (1) and (2), the many-body Hamiltonian of the weakly interacting Bose gas becomes diagonal in the $b_{\mathbf{q}}$'s, representing a system of free quasiparticles whose energy is given by the famous Bogoliubov dispersion law:

$$\epsilon(q) = \left[q^2 c^2 + \left(\frac{q^2}{2m} \right)^2 \right]^{1/2}. \quad (3)$$

In Eq. (3), $c = [gn/m]^{1/2}$ is the sound velocity fixed by the density of the gas n and by the parameter g characterizing the interaction term $g \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$ of the many-body Hamiltonian. The interaction parameter g is determined by the s -wave scattering length a through the relation $g = 4\pi\hbar^2 a/m$. The dispersion law (3) fixes the value of the quasiparticle amplitudes u_q and v_q , which can be written as

$$u_q, v_q = \pm \frac{\epsilon(q) \pm q^2/2m}{2\sqrt{\epsilon(q)q^2/2m}}; \quad (4)$$

and satisfy the normalization condition $u_q^2 - v_q^2 = 1$. At low momentum transfer ($q^2/2m \ll mc^2$) the Bogoliubov excitations are phonons characterized by the linear disper-

sion law $\epsilon = qc$ and the amplitudes u_q and v_q exhibit the infrared divergence $u_q \sim -v_q \sim (mc/2q)^{1/2}$. Vice versa, at high momentum transfer the dispersion law (3) approaches the free energy $q^2/2m$ and the Bogoliubov amplitudes take the ideal gas values $u_q = 1$, $v_q = 0$.

Bogoliubov's theory has been developed also for nonuniform gases. The theory has proved to be an excellent approximation for trapped Bose-Einstein condensates of alkali atoms. In particular, it has been successfully used to interpret the available experimental results on the propagation of phonons, namely, the excitation of the lowest frequency modes [2,3], corresponding to discretized phonon oscillations of the system [4,5], the generation of wave packets propagating in the medium with the speed of sound [6], and the excitation of phonons through inelastic photon scattering [7]. In the latter cases, the dispersion law (3) can be defined locally through the density dependence of the sound velocity. All these experiments, however, reveal the propagation of phonons only in coordinate space, where the equations of motion take the classical hydrodynamic form, and not in momentum space, where Bogoliubov's transformations (1) and (2) exhibit their peculiar character.

In this work we suggest a procedure to measure the Bogoliubov parameters u_q and v_q in a trapped Bose-Einstein condensed gas. Our strategy is based on the following two steps:

(A) First, one generates a collection of quasiparticles in the sample by means of the technique already used in [7]. This is based on an inelastic collisional process (two photon Bragg scattering) which can be implemented with two detuned lasers transferring momentum \mathbf{q} and energy $\hbar\omega$ to the sample. Here $\mathbf{q} = \hbar(\mathbf{k}_1 - \mathbf{k}_2)$ and $\omega = (\omega_1 - \omega_2)$ are fixed by the difference of the wave vectors and the corresponding frequencies of the two lasers. In order to excite quasiparticles in the phonon regime one should satisfy the condition $q < mc$. Let us call N_{ph} the number of quasiparticles with momentum \mathbf{q} generated by this first Bragg pulse and let us assume, for simplicity, that the

system can be treated as a uniform gas. According to the Bogoliubov transformations (1) and (2), the momentum distribution of the gas will be modified as

$$n(\mathbf{p}) = n_0(\mathbf{p}) + N_{\text{ph}}[u_q^2 \delta(\mathbf{p} - \mathbf{q}) + v_q^2 \delta(\mathbf{p} + \mathbf{q})], \quad (5)$$

where $n_0(\mathbf{p})$ is the momentum distribution at equilibrium and δ is the usual δ function. Equation (5) reveals the occurrence of two new terms describing particles propagating with directions parallel and antiparallel to the momentum \mathbf{q} of the quasiparticles (hereafter called phonons) and weights proportional, respectively, to u_q^2 and v_q^2 . The total momentum, $\mathbf{P} = \int d\mathbf{p} \mathbf{p} n(\mathbf{p})$ carried by the system, is equal to $\mathbf{q}N_{\text{ph}}$, as a result of the normalization condition $u_q^2 - v_q^2 = 1$.

(B) In the second step of the experiment one measures the momentum distribution (5) by sending a second Bragg pulse immediately after the first Bragg pulse. The momentum \mathbf{Q} , and the energy $\hbar\Omega$ transferred by the second pulse should be much larger than the ones of the first pulse since, in order to be sensitive to the momentum distribution of the sample, the scattering should probe the individual motion of particles [8,9]. More precisely, one must satisfy the condition $\hbar\Omega \sim Q^2/2m \gg mc^2$. The measured quantity is the dynamic structure factor which, in the large Q regime, takes the form [10]

$$S(Q, \Omega) = \frac{m}{Q} \int dp_x dp_y n(p_x, p_y, p_z), \quad (6)$$

where $p_z = m(\hbar\Omega - Q^2/2m)/Q$ and we have assumed \mathbf{Q} to be directed along the \mathbf{z} axis. By inserting (5) into (6), one finds that the dynamic structure factor exhibits, in addition to the original peak located at $\Omega = Q^2/2m\hbar$, two side peaks at

$$\Omega_{\pm} = \frac{Q^2}{2m\hbar} \pm \frac{\mathbf{q} \cdot \mathbf{Q}}{m\hbar}. \quad (7)$$

By denoting with S_+ and S_- their contributions to the integrated strength $\int d\Omega S(Q, \Omega) = N$, one finds $S_+ = N_{\text{ph}}u_q^2$ and $S_- = N_{\text{ph}}v_q^2$ or, equivalently, $N_{\text{ph}} = S_+ - S_-$ and $v_q^2 = S_-/(S_+ - S_-)$. If the quantity $S_+ + S_- = N_{\text{ph}}(u_q^2 + v_q^2)$ is much smaller than N , the normalization of the central peak remains close to the unperturbed value N . From the above discussion one concludes that the measurement of the dynamic structure factor at high momentum transfer \mathbf{Q} and, in particular, of the two strengths S_{\pm} would provide direct access to the number of phonons generated with the first Bragg pulse, as well as to the value of the corresponding quasiparticle amplitudes.

Expression (6) for the dynamic structure factor ignores the effects of the final state interactions which are responsible for both the line shift of the curve $S(Q, \Omega)$, as a function of Ω , and for its broadening. These effects can be safely calculated within Bogoliubov's theory and, in the large Q domain, are both fixed by the chemical potential of the gas [8,9]. The broadening due to mean field effects

should not be confused with the Doppler broadening included in Eq. (6). The latter is due to the fact that, even in the equilibrium configuration, the momentum distribution of a condensate of axial size R_z has a width $\sim \hbar/R_z$. The width of $S(Q, \Omega)$ is then proportional to $\hbar Q/R_z$. In the following, we will consider highly elongated condensates so that the relevant values of q are always larger than \hbar/R_z and the Doppler broadening, due to the finite size of the system, can be ignored. For a safe identification of the two phonon peaks (7) and of the corresponding strengths S_{\pm} it is crucial that the separation $\Delta\Omega = \pm(\mathbf{q} \cdot \mathbf{Q})/m\hbar$ between the phonon and central peaks be larger than the mean-field effect. This imposes the condition

$$qQ/m > \mu, \quad (8)$$

where we have chosen the two vectors \mathbf{q} and \mathbf{Q} parallel in order to maximize the separation $\Delta\Omega$. Equation (8), together with the condition $q \gg \hbar/R_z$, shows that the momentum q of the phonons generated by the first Bragg pulse should not be too small.

In the second part of the work we explore in detail the microscopic mechanisms of generation of phonons produced by the first Bragg pulse, taking into account the fact that our system is nonuniform and that the time duration of the pulse is finite. We consider a gas of interacting atoms initially confined by a harmonic potential of the form $V_{\text{ho}}(x, y, z) = m[\omega_{\perp}^2(x^2 + y^2) + \omega_z^2 z^2]/2$. The generation of phonons is analyzed through the numerical solution of the time dependent Gross-Pitaevskii equation for the order parameter $\Psi(\mathbf{r}, t)$ [5] in the presence of the additional external potential

$$V_{\text{Bragg}}(z, t) = Vf(t) \cos(qz/\hbar - \omega t), \quad (9)$$

which reproduces the effects of the inelastic scattering associated with the two photon Bragg pulses directed along the axial z direction (see, for example, Ref. [11]). In Eq. (9) the parameter V is the strength of the Bragg pulse while the envelope function $f(t)$ was chosen of the form $f(t) = \frac{1}{2}[1 + \tanh(t/t_{\text{up}})]$, and $f(t) = 0$ for $t > t_B$. Here t_B is the duration of the Bragg pulse, while t_{up} fixes its rise time. By varying the values of q and ω of the perturbation (9) different regions of the nonuniform gas are excited with different intensity. In fact, according to the Bogoliubov dispersion (3), the condensate is in resonance with the periodic perturbation $\cos(qz/\hbar - \omega t)$ for values of the density satisfying the condition $c^2 = gn/m = [\hbar^2 \omega^2 - (q^2/2m)^2]/q^2$.

The ground state, corresponding to the stationary solution of the Gross-Pitaevskii equation at large negative times t , was obtained by means of the steepest descent method [12]. For the time dependent solutions we have used a numerical code developed in Ref. [13], suitable for axially symmetric condensates. The parameter V has been chosen in order to generate a number of phonons corresponding to 5%–10% of the total number of atoms. In this way, one produces a visible bunch of excitations whose features

can be still described using linear response theory. Higher values of V were also considered to explore nonlinear effects. The Bragg pulse duration t_B was always taken to be significantly less than the oscillation time in the axial direction. This requirement is needed in order to relate the total momentum transferred by the photons with the actual momentum carried by the system at the end of the first Bragg pulse, thereby ignoring the effects of the external force produced by the harmonic potential during the pulse. This condition is well satisfied in the experiment of Ref. [7] where the total momentum P_z of the condensate was measured after the Bragg pulse.

An example of the density $|\Psi(\mathbf{r}, t_B)|^2$ as a function of z and for $r_\perp = [x^2 + y^2]^{1/2} = 0$, is shown in Fig. 1. The condensate in this figure has $N = 6 \times 10^7$ sodium atoms confined in a trap with $\omega_\perp = 2\pi \times 150$ Hz and $\omega_z = 0.12\omega_\perp$. This corresponds to a Thomas-Fermi parameter $Na/a_\perp = 10\,000$, where $a_\perp = (\hbar/m\omega_\perp)^{1/2}$. We have chosen a duration of the Bragg pulse $t_B = 0.25 \times 2\pi/\omega_\perp$ (~ 1.7 ms) and intensity $V = 1.25\hbar\omega_\perp$. The values of q and ω are $q = 1\hbar/a_\perp$ and $\omega = 4.13\omega_\perp$. With these parameters we are close to the phonon regime ($q^2/2m = 0.02mc^2$).

In Fig. 2 we give the corresponding prediction for the dynamic structure factor $S(Q, \Omega)$, measurable with the second Bragg pulse. This quantity is evaluated in impulse approximation and is determined by the longitudinal momentum distribution, as in Eq. (6),

$$\int dp_x dp_y n(p_x, p_y, p_z) = \int dx' dy' dz' dz e^{-ip_z(z-z')/\hbar} \times \Psi^*(x', y', z, t_B) \times \Psi(x', y', z', t_B), \quad (10)$$

where $p_z = m(\hbar\Omega - Q^2/2m)/Q$. Final state interaction effects are ignored in this approximation, but they do not affect the conclusions of our analysis provided condition (8) is satisfied.

Figure 2 clearly shows the appearance of the two peaks in $S(Q, \Omega)$ at the frequencies predicted by Eq. (7). The

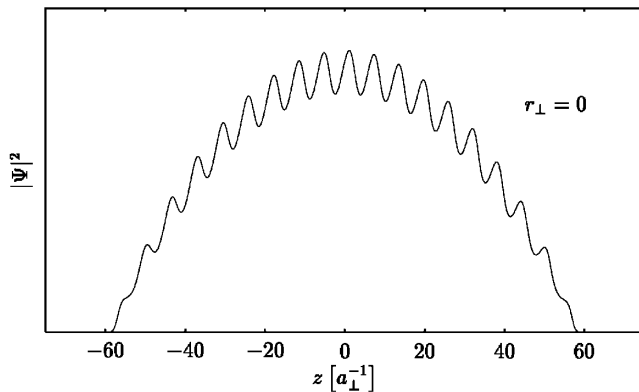


FIG. 1. Density profile of the condensate as a function of z evaluated at $r_\perp = 0$ after the first Bragg pulse.

difference $S_+ - S_-$ between their strengths gives the number of phonons N_{ph} ; this turns out to be $\sim 6 \times 10^6$, i.e., about 10% of the total number of atoms. We have verified that the results are independent of the choice of the rise time of the pulse, provided $t_{\text{up}} \leq 0.05 \, 2\pi/\omega_\perp$. We have also checked that the system responds in a linear way, by verifying that the value of P_z increases quadratically with V up to $V \sim 3\hbar\omega_\perp$; moreover, for sufficiently long times, the number of phonons generated by the pulse increases linearly with t_B as predicted by perturbation theory. We point out that condition (8), which ensures the visibility of the two peaks in $S(Q, \Omega)$, can be satisfied with reasonable choices of the momentum Q of the second Bragg pulse. Taking, for example, the value $Q = 21 \, \mu\text{m}^{-1}$ [8], we get $Qq/m = 36\hbar\omega_\perp$, to be compared with the value $\mu = 25\hbar\omega_\perp$ of the chemical potential. It is finally worth noticing that, since each phonon carries momentum q , their number can also be obtained by measuring the total momentum P_z after the first Bragg pulse, as done in the experiment of Ref. [7]: $N_{\text{ph}} = P_z/q$. This is useful when $q \ll mc$, since in this case $S_+ \sim S_-$ and the difference $S_+ - S_-$ may be difficult to extract.

The strengths S_\pm can be used to estimate the value of v_q^2 . Our results are given in Fig. 3 as a function of the first Bragg pulse duration. The three curves have been obtained with different choices for the transferred energy and momentum, ω and q , but they correspond to the same resonant density, i.e., the Bragg pulse excites the system in resonance at the same density (~ 0.67 of the central value). The conditions for such resonant behavior have been taken from the local density approximation discussed in [9]. Because of the different values of q , the three curves correspond also to different values of v_q^2 , since this quantity depends on q and on the density through the ratio mgn/q^2 , as predicted by Eqs. (3) and (4). Our results clearly show this effect. In order to make the analysis more quantitative, we also report, for each curve, the value predicted by Eq. (4) with $\epsilon(q) = \hbar\omega$. In the case of a periodic perturbation in a uniform gas the calculated curves of v_q should

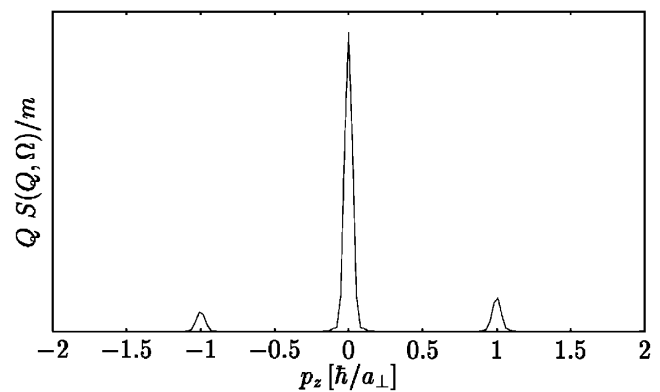


FIG. 2. Dynamic structure factor as a function of $p_z = m(\hbar\Omega - Q^2/2m)/Q$ after the first Bragg pulse.

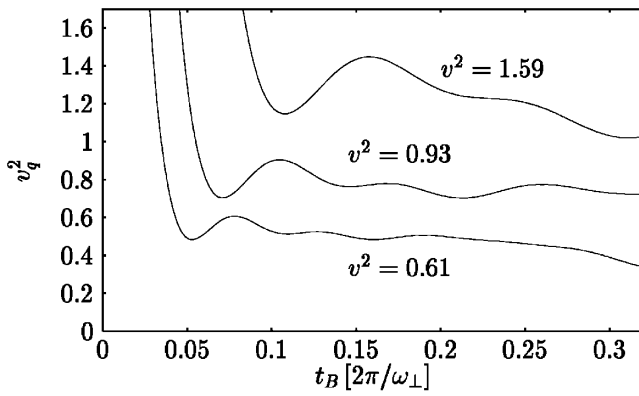


FIG. 3. v_q^2 as a function of time for different choices of q (in units of \hbar/a_\perp) and ω (in units of ω_\perp). In order, from top to bottom: $q = 1$, $\omega = 4.13$; $q = 1.5$, $\omega = 6.25$, and $q = 2$, $\omega = 8.44$. The values predicted by Eq. (4) are also reported.

coincide with prediction (4). In our calculations we find that the values of v_q^2 exhibit oscillations with frequency 2ω and a slight decrease as a function of time. The behavior of v_q^2 at short times is the consequence of the high frequency components contained in the Fourier transform of the Bragg potential (9), whose effects cannot be simply described employing a local density picture. The decrease of the signal at larger times is probably the consequence of the diffusion of phonons towards regions of lower density as well as of nonlinear effects. Despite these effects, Fig. 3 clearly reveals the important features predicted by Bogoliubov theory for the quasiparticle amplitude v and, in particular, its dependence on the relevant parameters of the system.

The results of Figs. 1–3 refer to conditions of linear or almost linear regime. It is also interesting to explore the response of the condensate to a highly nonlinear perturbation generating a number of excitations comparable to the total number of atoms. This can be achieved by increasing the strength V of the Bragg pulse. In Fig. 4 we show the dynamic structure factor $S(Q, \Omega)$ calculated after the first Bragg pulse, in conditions of high nonlinearity ($V = 25\hbar\omega_\perp$). Remarkably, the $p_z = 0$ peak, corresponding to the initial condensate, has almost disappeared. In this case, the appearance of additional peaks, associated with the second and third harmonics $p_z = \pm 2q$ and $p_z = \pm 3q$ in the longitudinal momentum distribution, is clearly visible. Obviously, the connection between the structure of these side peaks and the quasiparticle amplitudes u and v is less direct than in the linear regime. Further analysis is under way, also in view of future experiments.

In conclusion, we have suggested an experimental method to measure Bogoliubov's quasiparticle amplitudes in a trapped Bose gas at low temperature. In such an

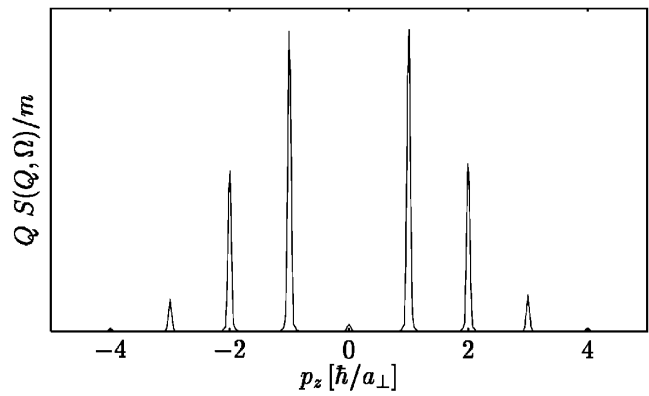


FIG. 4. Same as Fig. 2. Here V is large in order to drive the system out of the linear regime.

experiment, the condensate is hit by a sequence of two Bragg pulses. The first (low q momentum transfer) pulse generates a collection of phonons which are subsequently mapped in momentum space by the second (high Q) pulse. A numerical simulation has allowed us to test our predictions and to show that our proposal is compatible with the presently available experimental possibilities. This experiment would provide the first direct measurement of the Bogoliubov quasiparticle amplitudes, which are of fundamental importance in the theory of Bose-Einstein condensation.

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