

Violations of Local Realism by Two Entangled N -Dimensional Systems Are Stronger than for Two Qubits

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Tests of local realism versus quantum mechanics based on Bell's inequality employ two entangled qubits. We investigate the general case of two entangled quantum systems defined in N -dimensional Hilbert spaces, or "quNits." Via a numerical linear optimization method we show that violations of local realism are stronger for two maximally entangled quNits ($3 \leq N \leq 9$) than for two qubits and that they increase with N . The two quNit measurements can be experimentally realized using entangled photons and unbiased multipoint beam splitters.

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Bell [1] has shown that no local realistic models can agree with all quantum mechanical predictions for the maximally entangled states of two two-state systems (qubits). After some years researchers started to ask questions about the Bell theorem for more complicated systems. The most surprising answer came from the Greenberger-Horne-Zeilinger (GHZ) theorem [2]: for three or more qubits the conflict between local realism and quantum mechanics is much sharper than for two qubits. The other possible extensions are entangled states of pairs of N -state systems, "quNits," with $N \geq 3$. First results, in 1980–1982, suggested that the conflict between local realism and quantum mechanics diminishes with growing N [3]. This was felt to be in concurrence with the old quantum wisdom of higher quantum numbers leading to a quasi-classical behavior. However, that early research was confined to Stern-Gerlach-type measurements performed on pairs of entangled $\frac{N-1}{2}$ spins [3]. Since operation of a Stern-Gerlach device depends solely on the orientation of the quantization axis, i.e., on only two parameters, devices of this kind cannot make projections into arbitrary states of the subsystems. That is, they cannot make full use of the richness of the N -dimensional Hilbert space.

In the early 1990's Peres and Gisin [4] considered certain *dichotomic* observables applied to maximally entangled pairs of quNits. They showed that the violation of local realism, or more precisely of the Clauser-Horne-Shimony-Holt [1] inequalities, survives the limit of $N \rightarrow \infty$, but never exceeds the violation by two qubits, in agreement with the Cirel'son limit [5], i.e., it is limited by the factor of $\sqrt{2}$. Therefore, the question whether the violation of local realism increases or not with growing N for general observables was still left open.

To answer this question it is necessary first to adopt an objective measure of the magnitude of violation of local realism. To this end, consider two quNit systems described by mixed states in the form of

$$\rho_N(F_N) = F_N \rho_{\text{noise}} + (1 - F_N) |\Psi_{\text{max}}^N\rangle \langle \Psi_{\text{max}}^N|, \quad (1)$$

where the positive parameter $F_N \leq 1$ determines the "noise fraction" within the full state, $\rho_{\text{noise}} = \frac{1}{N^2} \hat{I}$, and $|\Psi_{\text{max}}^N\rangle$ is a maximally entangled two quNit state, say

$$|\Psi_{\text{max}}^N\rangle = \frac{1}{\sqrt{N}} \sum_{m=1}^N |m\rangle_A |m\rangle_B. \quad (2)$$

In (2) $|m\rangle_A$ ($|m\rangle_B$) describes particle A (B) in its mode m . One has ${}_x \langle m | m' \rangle_x = \delta_{m,m'}$, with $x = A, B$. The threshold maximal F_N^{max} , for which the state $\rho_N(F_N)$ still does not allow a local realistic model, will be our value of the strength of violation of local realism. The higher F_N^{max} the higher noise admixture will be required to hide the non-classicality of the quantum prediction. In experiments the visibility parameter V , effectively equivalent to $1 - F_N$, is the usual measure of the reduction of interferometric contrast (visibility).

We shall study the case of two observers Alice and Bob performing measurements of local nondegenerate observables, each on her/his quNit of an entangled pair in the state $\rho_N(F_N)$. Let us imagine that Alice can choose between two nondegenerate observables A_1 and A_2 , and that each observable is defined such that it has the full spectrum characterized by all integers from $k = 1$ to N . Bob can choose between B_1 and B_2 , both with the same spectrum as above ($l = 1, 2, \dots, N$). Thus, the observers can perform 2×2 mutually exclusive global experiments. The quantum probability for the specific pair of results, k for Alice and l for Bob, provided a specific pair of local observables is chosen, A_i by Alice and B_j by Bob, will be denoted by $P_{F_N}^{QM}(k; l | A_i, B_j)$. Quantum mechanics makes predictions for the complete set of $4N^2$ such probabilities, and nothing more.

The hypothesis of local hidden variables tries to go beyond. The basic assumption there is that each particle carries a probabilistic or deterministic set of instructions how to respond to all possible local measurements it might be subject to. Therefore local realism assumes the existence of non-negative joint probabilities involving all possible observations from which it should be possible to

obtain all the quantum predictions as marginals (see, e.g., [6,7]). Let us denote these hypothetical probabilities by $P^{HV}(k, m; l, n | A_1, A_2, B_1, B_2)$, where k and m represent the outcome values for Alice's observables (l and n for Bob's). In quantum mechanics one cannot even define such objects, since they involve mutually incompatible measurements. The local hidden variable probabilities for the experimentally observed events, k (m) by Alice measuring A_1 (A_2), and l (n) by Bob measuring B_1 (B_2), are the marginals

$$\begin{aligned} P^{HV}(k; l | A_1, B_1) &= \sum_m \sum_n P^{HV}(k, m; l, n), \\ P^{HV}(k; n | A_1, B_2) &= \sum_m \sum_l P^{HV}(k, m; l, n), \\ P^{HV}(m; l | A_2, B_1) &= \sum_k \sum_n P^{HV}(k, m; l, n), \\ P^{HV}(m; n | A_2, B_2) &= \sum_k \sum_l P^{HV}(k, m; l, n), \end{aligned} \quad (3)$$

where $P^{HV}(k, m; l, n)$ is a shorthand notation for $P^{HV}(k, m; l, n | A_1, A_2, B_1, B_2)$. The $4N^2$ equations (3) form the full set of necessary and sufficient conditions for the existence of a local realistic description of the experiment, i.e., for the joint probability distribution $P^{HV}(k, m; l, n)$. The Bell theorem says that certain predictions by quantum mechanics are in conflict with the local hidden variable model (3). Evidently, the conflict disappears when enough noise is added, as in the state (1), since that noise has a local realistic model. Therefore a threshold F_N^{\max} exists below which one cannot have any local realistic model with $P^{HV}(k; l | A_i, B_j) = P_{F_N}^{QM}(k; l | A_i, B_j)$. Our goal is to find observables for the two quNits returning the highest possible critical F_N^{\max} .

Up to date, no one has derived Bell-type inequalities that are necessary and sufficient conditions for (3) to hold, with the exception of the $N = 2$ case (see [7]). However there are numerical tools, in the form of the very well-developed theory and methods of linear optimization, which are perfectly suited for tackling exactly such problems directly, without any Bell inequalities [8].

The quantum probabilities, when the state is given by (1), have the following structure:

$$\begin{aligned} P_{F_N}^{QM}(k; l | A_i, B_j) &= \frac{1}{N^2} F_N \\ &+ (1 - F_N) P^{QM}(k; l | A_i, B_j), \end{aligned} \quad (4)$$

where $P^{QM}(k; l | A_i, B_j)$ is the probability for the given pair of events for the pure maximally entangled state. The set of conditions (3) with $P_{F_N}^{QM}(k; l | A_i, B_j)$ replacing $P^{HV}(k; l | A_i, B_j)$ imposes linear constraints on the N^4

“hidden probabilities” $P^{HV}(k, m; l, n)$ and on the parameter F_N , which are the non-negative unknowns. We have more unknowns ($N^4 + 1$) than equations ($4N^2 + 1$, with the normalization condition for the hidden probabilities), and we want to find the minimal F_N for which the set of constraints can still be satisfied. This is a typical linear optimization problem for which lots of excellent algorithms exist. We have used the state-of-the-art algorithm HOPDM 2.30 (higher order primal dual method) [9]. It is important to stress that for cross checking four independently written codes were used, one of them employing a different linear optimization procedure (from the NAG Library).

We were interested in finding such observables for which the threshold F_N acquires the highest possible value. To find optimal sets of observables we have used a numerical procedure based on the downhill simplex method (so-called amoeba) [10]. If the dimension of the domain of a function is D (in our case $D = 4n$, where n is the number of parameters specifying the nondegenerate local observables belonging to a chosen family), the procedure first randomly generates $D + 1$ points. In this way it creates the vertices of a starting simplex. Next it calculates the value of the function at the vertices and starts exploring the space by stretching and contracting the simplex. In every step, when it finds vertices where the value of the function is higher than in others, it “goes” in this direction (see, e.g., [10]).

Let us now move to the question of finding a family of observables, which returns critical F_N 's that are above the well-known threshold for the two qubit case, $1 - \frac{1}{\sqrt{2}}$. As was said earlier, and was confirmed by our numerical results, Stern-Gerlach-type measurements are not suitable. More exotic observables are needed.

First we discuss how experiments on two entangled quNits might be performed. In view of the unavailability of higher spin entanglement it is fortunate that quNit entanglement can be studied exploiting momentum conservation in the many processes of the two-particle generation, most notably in the parametric down-conversion generation of entangled photon pairs. This results in strong correlations

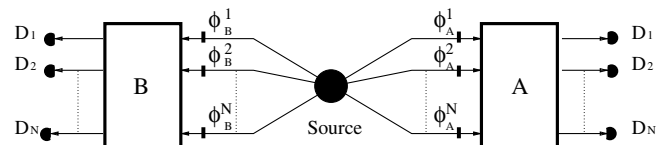


FIG. 1. The experiment of Alice and Bob with entangled quNits. Each of their measuring apparatuses consist of a set of N phase shifters just in front of a $2N$ port Bell multiport, and N photon detectors D_k, D_l (perfect, in the gedanken situation described here) which register photons in the output ports of the device. The phase shifters serve the role of the devices which set the free macroscopic, classical parameters that can be controlled by the experimenters. The source produces a beam-entangled two-particle state.

between the propagation directions of the particles in a pair. One can then submit N spatial modes of each particle to a multiport beam splitter [11].

Application of multiports in the context of quantum entanglement has been first discussed by Klyshko [12]. Proposals of Bell experiments with the multiports were presented in [13], and further developed in [11]. Multiport devices can reproduce all finite dimensional unitary transformations for single-photon states [14], therefore they are characterized by $N^2 - 1$ real parameters.

In order to limit computer time we restricted our analysis to unbiased multiports [11], more specifically to Bell multiports. Unbiased multiports have the property that a photon entering into any *single* input port (out of the N), has equal chances to exit from any output port. In addition, for Bell multiports [11] the elements of their unitary transition matrix, \mathbf{U}^N , are *solely* powers of the N th root of unity $\gamma_N = \exp(i2\pi/N)$, namely, $U_{ji}^N = \frac{1}{\sqrt{N}}\gamma_N^{(j-1)(i-1)}$.

Let us now imagine two spatially separated experimenters who perform the experiment of Fig. 1 (described in the caption). The initial maximally entangled state (2) of the two quNits can be prepared with the aid of parametric down-conversion (see [11]). The two sets of phase shifters at the inputs of the multiports (one phase shifter in each beam) introduce phase factor $e^{i(\phi_A^m + \phi_B^m)}$ in front of the m th component of the state (2), where ϕ_A^m and ϕ_B^m denote the local phase shifts.

Each set of local phase shifts constitutes the interferometric realizations of the “knobs” at the disposal of the observer controlling the local measuring apparatus, which incorporates also the Bell multiport and N detectors. In this way the local observable is defined. Its eigenvalues refer simply to registration at one of the N detectors behind the multiport. The quantum prediction for the joint probability $P_{F_N}^{QM}(k, l)$ to detect a photon at the k th output of the multiport A and another one at the l th output of the multiport B is given by [11]

$$P_{F_N}^{QM}(k, l; \phi_A^1, \dots, \phi_A^N, \phi_B^1, \dots, \phi_B^N) = \frac{F_N}{N^2} + \frac{1 - F_N}{N} \left| \sum_{m=1}^N \exp[i(\phi_A^m + \phi_B^m)] U_{mk}^N U_{ml}^N \right|^2$$

$$= \left(\frac{1}{N^3} \right) \left[N + 2(1 - F_N) \sum_{m>n}^N \cos(\Phi_{kl}^m - \Phi_{kl}^n) \right], \tag{5}$$

where $\Phi_{kl}^m \equiv \phi_A^m + \phi_B^m + [m(k + l - 2)]\frac{2\pi}{N}$. The counts at a single detector, of course, are constant, and do not depend upon the local phase settings: $P_{F_N}^{QM}(k) = P_{F_N}^{QM}(l) = 1/N$.

The numerical values of the threshold F_N are given in Fig. 2. It is evident that indeed two entangled quNits violate local realism stronger than two entangled qubits, and that the violation increases monotonically with N . It is tempting to contemplate the limit of $N \rightarrow \infty$. While obviously the values of F_N^{\max} seem to saturate, at present we cannot give a definite asymptotic value.

A few words of comment are needed. One may argue that because of the rather large number of local macroscopic parameters (the phases) defining the function to be maximized with the amoeba we could have missed the global minimum. While this argument cannot be ruled out in principle, we stress that in that case the ultimate violation would even be larger. This would only strengthen our conclusion that two entangled quNits are in stronger conflict with local realism than two entangled qubits.

Based on the numerical results, i.e., the values of the optimal phase settings, and on the structure of the local hidden variable model for F_3^{\max} , an algebraic calculation was performed [15] showing that for the two qutrits ($N = 3$) experiment the exact value for F_3^{\max} is $\frac{11-6\sqrt{3}}{2}$. One should also mention that for two spin 1 particles in a singlet state observed by two Stern-Gerlach apparatuses our method gives $F_3^{SG} = 0.1945$, which is much smaller than $1 - \frac{1}{\sqrt{2}}$, confirming that such measurements are not optimal in the

sense of leading to maximal possible violations of local realism.

An important question is whether unbiased Bell multiports provide us with a family of observables in maximal conflict with local realism. For a check of this question we have also calculated the threshold value of F_3 for the case where both observers apply to the incoming qutrit the most

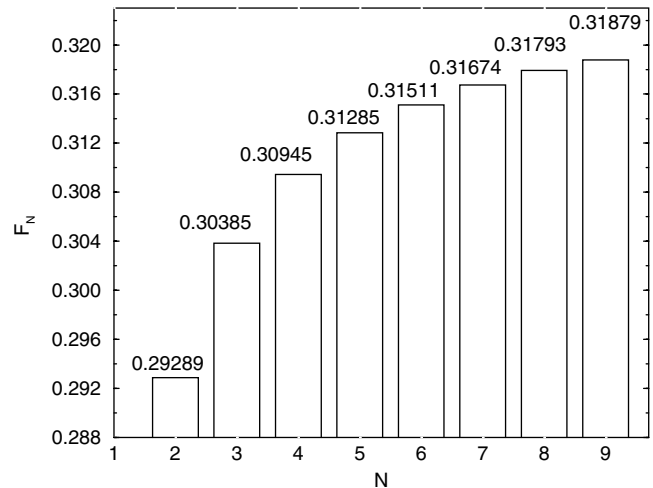


FIG. 2. Maximal fraction F_N^{\max} of pure noise admixture to a maximally entangled two quNite system, such that a local realistic explanation still cannot be upheld. For smaller noise fractions a conflict arises between quantum mechanics and local realism. The result for $N = 2$ agrees with the standard threshold of $1 - \frac{1}{\sqrt{2}}$.

general unitary transformation belonging to a full SU(3) group (i.e., we have *any* trichotomic observables on each side). Again we have assumed that each observer chooses between two sets of local settings. However, in this case each set consists of eight local settings rather than the three (effectively two) in the tritter case. The result appears to be the same as for two tritters. While this might suggest that for $N = 3$ Bell multiports are optimal devices to test quantum mechanics against local realism, this needs to be further investigated.

It is interesting to compare our results with the limit for the nonseparability of the density matrices (1). The critical minimal F_N for which a density matrix (1) is separable is $\frac{N}{N+1}$ (see [16]). The fact that this limit is always higher than ours indicates that the requirement of having a local quantum description of the two subsystems is a much more stringent condition than the requirement of admitting any possible local realistic model.

It will be interesting to consider within our approach different families of states, generalizations to more than two particles, extensions of the families of observables, and to see if more than two (e.g., A_1, A_2, A_3) experiments performed on either side can lead to even stronger violations of local realism. The questions concerning the critical F_N are also important in the attempts to generalize Ekert's quantum cryptographic protocol to qutrits and higher systems [17].

Our method is numerical and is based on linear optimization. It is a development of the approach of [8]. The exploding (with N) difficulty of approaching this type of problems via algebraic-analytical methods (generalized Bell inequalities, via the Farkas lemma, etc.) has been exposed in [7].

It will certainly be fascinating to see laboratory realizations of the experimental schemes discussed here.

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