Catalysis of Entanglement Manipulation for Mixed States

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We consider entanglement-assisted remote quantum state manipulation of bipartite mixed states. Several aspects are addressed: we present a class of mixed states of rank two that can be transformed into another class of mixed states under entanglement-assisted local operations with classical communication, but for which such a transformation is impossible without assistance. Furthermore, we demonstrate enhancement of the efficiency of purification protocols with the help of entanglement-assisted operations. Finally, transformations from one mixed state to mixed target states which are sufficiently close to the source state are contrasted with similar transformations in the pure-state case.

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Entanglement between spatially separated quantum systems has important implications on fundamental issues of quantum mechanics and forms the basis for most of the practical applications of quantum information theory [1,2]. In many of these applications two or more parties have direct access to only parts of a composite quantum system, but may communicate by classical means and may thereby coordinate their actions. In light of recent progress in quantum information theory, entanglement is often viewed as the essential resource for processing and transmitting quantum information.

As has been demonstrated in Ref. [3], entanglement is indeed an intriguing type of resource: the mere presence of entanglement can be an advantage when the task is to transform an initial state into a certain final state with the use of local quantum operations and classical communication (LOCC). There are indeed target states which cannot be reached by LOCC starting from a particular initial state, but which can be reached with the assistance of a distributed pair of auxiliary quantum systems in a particular known state, even though these auxiliary quantum systems are left in *exactly the same state.* Such transformations are called entanglement-assisted LOCC (ELOCC) operations.

This phenomenon is quite remarkable as the entanglement which serves as a "catalyst" for the otherwise forbidden "reaction" is not consumed. The basis of the example given in Ref. [3] is a criterion presented in Ref. [4]: A joint pure state corresponding to $|\psi\rangle$ can be transformed into another $|\phi\rangle$ with the use of LOCC if, and only if, the set of ordered Schmidt coefficients characterizing the initial state is majorized [5] by the set of ordered Schmidt coefficients of the final state. Curiously, it is the strange class of ELOCC operations that adds a new flavor to the initial question raised in Ref. [4]; "What tasks may be accomplished using a given physical resource?" The class of ELOCC operations is in fact more powerful than LOCC even without a concomitant consumption of the physical resource entanglement [3,6].

In practical applications, one would expect to always deal with entangled mixed states rather than with pure states. Unfortunately, such a convenient tool as the majorization criterion is missing in the mixed-state case, and the question whether a particular entanglement transformation from one mixed state into another mixed state is possible seems to be much more involved [7]. In mixed quantum mechanical states both classical correlations and intrinsic quantum correlations may be present, which makes the structure of mixed-state entanglement a more complex matter. A different aspect of the same problem is the well-known fact that a representation of a mixed state in terms of pure states is not uniquely defined, and it is essentially this ambiguity that prohibits a straightforward application of the majorization criterion.

In this Letter we demonstrate that even for mixed states the set of tasks that can be accomplished with entanglement-assisted local operations is strictly larger than the set of tasks which may be performed with mere LOCC. This fact is not obvious *a priori*, bearing in mind that, e.g., pure states and mixed states behave very differently as far as purification is concerned [8]. The problem of catalysis of entanglement manipulation for mixed states will be approached as follows: (i) We give a class of mixed states of rank two that can be transformed into representatives of another class of mixed states with ELOCC but not with LOCC; (ii) we show that there are cases for which the proportion of a certain pure state in a mixture can be increased more efficiently with ELOCC operations than with sole LOCC; (iii) purification schemes are investigated for a practically important class of mixed states; (iv) "small transformations" in the interior of the state space are compared with similar entanglement manipulations in the pure-state case.

Definitions.—Let σ and ρ be states taken from the state space $\mathcal{S}(\mathcal{H})$ over $\mathcal{H}$, where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space associated with a bipartite quantum system consisting of parts A and B. We write, in the following, $\sigma \longrightarrow \rho$ under LOCC if σ can be transformed into ρ by applying local transformations and classical communication [4]. A pair of states ρ, σ is called incommensurate if both $\sigma \not\longrightarrow \rho$ and $\rho \not\longrightarrow \sigma$ under LOCC. For *pure* states σ

and ρ the (necessary and sufficient) majorization criterion for $\sigma \rightarrow \rho$ under LOCC reads as [4]

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} \beta_i \quad \text{for all } k = 1, \dots, N-1\,, \tag{1}$$

$N = \dim[\mathcal{H}_A] = \dim[\mathcal{H}_B]$, where $\alpha_1, \dots, \alpha_N$ and $\beta_1, \dots, \beta_N$ with $1 \geq \alpha_1 \geq \dots \geq \alpha_N \geq 0$ and $1 \geq \beta_1 \geq \dots \geq \beta_N \geq 0$ are the eigenvalues of $\mathrm{tr}_A[\sigma]$ and $\mathrm{tr}_A[\rho]$, respectively. Such a list is also referred to as an ordered list. The content of the conditions stated in Eq. (1) is abbreviated in the following as $\mathrm{tr}_A[\sigma] \prec \mathrm{tr}_A[\rho]$, with the majorization relation $\prec$ [5]. As for LOCC operations we use the notation $\sigma \rightarrow \rho$ under ELOCC, if

$$\sigma \otimes \omega \rightarrow \rho \otimes \omega \tag{2}$$

under LOCC for an appropriately chosen catalyst state ω [3]. This state ω is an entangled state of another bipartite quantum system. Note that in the course of the transformation this state remains fully unchanged.

Mixed-state catalysis of entanglement manipulation.—The first result concerns the existence of incommensurate genuinely mixed states such that, with the use of some appropriately chosen catalyst state, the initial state can be converted into the final state while fully retaining the catalyst state. That is, there exist mixed states $\sigma, \rho \in S(\mathcal{H})$ such that $\sigma \rightarrow \rho$ under ELOCC but not $\sigma \rightarrow \rho$ under LOCC. "Genuinely" mixed means here that the projections appearing in the spectral decomposition of the initial state cannot be locally distinguished. If this were possible the initial state would essentially be pure.

To see that mixed-state catalysis is possible we construct a class of states which exhibits this phenomenon. For this class of states the statement that $\sigma \rightarrow \rho$ under ELOCC follows immediately from the theorem presented in Ref. [4]. To prove that such a transformation is impossible under LOCC, the following Lemma is useful.

Lemma 1.—Let σ and ρ be mixed states of rank two of the form

$$\sigma = \lambda |\psi\rangle\langle\psi| + (1-\lambda)|\eta\rangle\langle\eta|\,, \tag{3a}$$

$$\rho = \mu |\phi\rangle\langle\phi| + (1-\mu)|\eta\rangle\langle\eta|\,, \tag{3b}$$

where $\mu = \lambda\, \mathrm{tr}[\chi]$,

$$\chi = \Pi|\psi\rangle\langle\psi|\Pi\,, \tag{4}$$

and $\Pi = 1 - |\eta\rangle\langle\eta|$. $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ are entangled pure states, while $|\eta\rangle\langle\eta|$ is a pure product state. Furthermore, $|\langle\eta\,|\,\phi\rangle|^2 = 0$. Then $\sigma \rightarrow \rho$ under LOCC implies that

$$\frac{\mathrm{tr}_A[\chi]}{\mathrm{tr}[\chi]} \prec \mathrm{tr}_A[|\phi\rangle\langle\phi|]\,. \tag{5}$$

Proof: Assume that $\sigma \rightarrow \rho$ under LOCC. The set of LOCC operations is included in the set of separable operations [9,10], that is, completely positive and trace-preserving maps that can be written in the form $\sigma \mapsto \sum_i (A_i \otimes B_i)\sigma(A_i \otimes B_i)^\dagger$ with Kraus operators A_i, B_i, $i = 1, 2, \dots,$ acting in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, where the trace-preserving property manifests as $\sum_i A_i^\dagger A_i = 1$ and $\sum_i B_i^\dagger B_i = 1$. For each i, the image of σ is contained in the range of ρ,

$$(A_i \otimes B_i)\sigma(A_i \otimes B_i)^\dagger \in \mathrm{range}(\rho)\,. \tag{6}$$

Since there is only a single product vector included in the range of ρ (which then amounts to a best separable approximation in the sense of [11]), the state $|\psi\rangle\langle\psi|$ must be mapped on $\nu|\phi\rangle\langle\phi| + (1-\nu)|\eta\rangle\langle\eta|$, where $\nu = \mu/\lambda$. $\Pi(A_i \otimes B_i)|\psi\rangle = \Pi(A_i \otimes B_i)\Pi|\psi\rangle$ for all i, and, hence,

$$\begin{aligned} \nu &= \mathrm{tr}\Big[\Pi \sum_i (A_i \otimes B_i)|\psi\rangle\langle\psi|(A_i \otimes B_i)^\dagger \Pi\Big] \\ &= \mathrm{tr}\Big[\sum_i \Pi(A_i \otimes B_i)\chi(A_i \otimes B_i)^\dagger \Pi\Big] \leq \mathrm{tr}[\chi]\,. \end{aligned} \tag{7}$$

As $\mathrm{tr}[\chi] = \nu$, it follows that $\chi/\mathrm{tr}[\chi] \rightarrow |\phi\rangle\langle\phi|$ under LOCC, which in turn implies, by the theorem in Ref. [4], that $\mathrm{tr}_A[\chi]/\mathrm{tr}[\chi] \prec \mathrm{tr}_A[|\phi\rangle\langle\phi|]$. □

The following one-parameter classes of states of rank two provide an example of catalysis for mixed states. Take $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\mathcal{H}_A, \mathcal{H}_B = \mathrm{span}\{|1\rangle, \dots, |5\rangle\}$ and let

$$\sigma = \lambda|\psi\rangle\langle\psi| + (1-\lambda)|55\rangle\langle 55|\,, \tag{8a}$$

$$\rho = \mu|\phi\rangle\langle\phi| + (1-\mu)|55\rangle\langle 55|\,, \tag{8b}$$

with $\mu = 0.95\lambda$ and

$$\begin{aligned} |\psi\rangle = {} & \sqrt{0.38}\,|11\rangle + \sqrt{0.38}\,|22\rangle + \sqrt{0.095}\,|33\rangle \\ & + \sqrt{0.095}\,|44\rangle + \sqrt{0.05}\,|55\rangle\,, \end{aligned} \tag{9a}$$

$$|\phi\rangle = \sqrt{0.5}\,|11\rangle + \sqrt{0.25}\,|22\rangle + \sqrt{0.25}\,|33\rangle\,. \tag{9b}$$

These states are clearly included in the sets of states considered in Lemma 1. Moreover, the initial state σ is genuinely mixed.

From Lemma 1 it follows that $\sigma \not\rightarrow \rho$ under LOCC for all values of $\lambda \in (0,1]$, as $\chi/\mathrm{tr}[\chi] = |\varphi\rangle\langle\varphi|$, where

$$|\varphi\rangle = \sqrt{0.4}\,|11\rangle + \sqrt{0.4}\,|22\rangle + \sqrt{0.1}\,|33\rangle + \sqrt{0.1}\,|44\rangle \tag{10}$$

as in Ref. [3]. Hence,

$$\frac{\mathrm{tr}_A[\chi]}{\mathrm{tr}[\chi]} \not\prec \mathrm{tr}_A[|\phi\rangle\langle\phi|]\,, \tag{11}$$

and therefore, $\sigma \not\rightarrow \rho$ under LOCC. However, it can be shown that $\sigma \rightarrow \rho$ under ELOCC. One may perform a local projective von Neumann measurement in system A associated with Kraus operators $A_1 = \sum_{i=1}^4 |ii\rangle\langle ii|$ and $A_2 = |55\rangle\langle 55|$ satisfying $A_1^\dagger A_1 + A_2^\dagger A_2 = 1$ (compare also Ref. [12]). If one gets the outcome corresponding to A_2, no further operations are applied. In the other case the final state is the pure state $|\varphi\rangle\langle\varphi|$ given by Eq. (10). As in Ref. [3] this state can be transformed into $|\phi\rangle\langle\phi|$ with the help of the catalyst state $\omega = (\sqrt{0.4}\,|66\rangle + \sqrt{0.6}\,|77\rangle) \times (\sqrt{0.4}\,\langle 66| + \sqrt{0.6}\,\langle 77|)$ [13], since

$$\mathrm{tr}_A[|\varphi\rangle\langle\varphi| \otimes \omega] \prec \mathrm{tr}_A[|\phi\rangle\langle\phi| \otimes \omega]. \tag{12}$$

Finally, the classical information about the outcomes is discarded in order to achieve ρ. Hence, it turns out that $\sigma \to \rho$ under ELOCC but $\sigma \not\to \rho$ under LOCC.

Increasing the proportion of a pure state in a mixture.—The possibility of catalysis of entanglement manipulations has implications on the efficiency of the attempts to increase the quota of some entangled state $|\xi\rangle\langle\xi|$ in a mixed state σ by applying a trace-preserving operation. Indeed, such protocols can be more efficient when employing ELOCC rather than exclusively using LOCC. More precisely, there are (genuinely) mixed states σ and pure states $|\xi\rangle\langle\xi|$ with the property that the maximal average attainable value of the fidelity under ELOCC,

$$F_{\mathrm{ELOCC}}(\sigma, |\xi\rangle\langle\xi|) = \sup_{\rho \in S^{\sigma}_{\mathrm{ELOCC}}} \langle\xi|\rho|\xi\rangle, \tag{13}$$

is strictly larger than the maximal attainable fidelity under LOCC,

$$F_{\mathrm{LOCC}}(\sigma, |\xi\rangle\langle\xi|) = \sup_{\rho \in S^{\sigma}_{\mathrm{LOCC}}} \langle\xi|\rho|\xi\rangle. \tag{14}$$

Here, $S^{\sigma}_{\mathrm{LOCC}}$ and $S^{\sigma}_{\mathrm{ELOCC}}$ are the sets of states that can be reached by applying LOCC and ELOCC, respectively, on an initial state σ.

This statement can be proven by considering an initial state σ of the form specified in Eq. (8a) with

$$|\psi\rangle = \varepsilon(\sqrt{0.4}\,|11\rangle + \sqrt{0.4}\,|22\rangle + \sqrt{0.1}\,|33\rangle + \sqrt{0.1}\,|44\rangle) + \sqrt{1 - \varepsilon^2}\,|55\rangle, \tag{15}$$

and one may choose $|\xi\rangle = |\phi\rangle$ as in Eq. (9b). Clearly,

$$\begin{aligned} F_{\mathrm{LOCC}}(\sigma, |\phi\rangle\langle\phi|) \leq{}& (1 - \lambda) F_{\mathrm{LOCC}}(|55\rangle\langle 55|, |\phi\rangle\langle\phi|) \\ &+ \lambda F_{\mathrm{LOCC}}(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|), \end{aligned} \tag{16}$$

as the components of the initial state σ are not locally distinguishable, and since the achievable fidelity can be no better than the sum of both best possible fidelities of each contribution. Under LOCC all separable states are accessible starting from $|55\rangle\langle 55|$. The (not necessarily pure) separable state closest to $|\phi\rangle\langle\phi|$ with respect to the fidelity is given by $|11\rangle\langle 11|$, and, therefore, $F_{\mathrm{LOCC}}(|55\rangle\langle 55|, |\phi\rangle\langle\phi|) = 1/2$. Finally, from $F_{\mathrm{ELOCC}}(\sigma, |\phi\rangle\langle\phi|) \geq \lambda\varepsilon^2 + (1 - \lambda\varepsilon^2)/2$, it follows that

$$F_{\mathrm{ELOCC}}(\sigma, |\phi\rangle\langle\phi|) > F_{\mathrm{LOCC}}(\sigma, |\phi\rangle\langle\phi|) \tag{17}$$

certainly holds for all $\varepsilon \in (\tilde{\varepsilon}, 1]$, with an appropriate $\tilde{\varepsilon} \in (0,1)$, independent of $\lambda \in (0,1)$, and for all $\varepsilon < 1$ the initial state is also genuinely mixed.

Purification procedures.—The previous two results unambiguously indicate that the class of ELOCC operations is more powerful than LOCC operations, not only on the subset of the boundary of $S(\mathcal{H})$ comprising the pure states but also in the interior of the set $S(\mathcal{H})$. Albeit this fact suggests that the use of supplementary catalyst states opens up possibilities to enhance purification procedures, ELOCC does not necessarily imply an improved efficiency in practically motivated problems. Consider the class of states studied in Ref. [8],

$$\sigma = \lambda|\psi\rangle\langle\psi| + (1 - \lambda)\zeta, \tag{18}$$

with the property that there exists a $\lambda_0 \in (0,1)$ such that σ is a separable state and that every state with a larger weight of $|\psi\rangle\langle\psi|$ is entangled. Furthermore, it is assumed that $\langle\psi|\zeta|\psi\rangle = 0$. This class of states includes the class of states consisting of a mixture of some pure state and the complete mixture in the corresponding state space, which is of salient importance in practical applications. In Ref. [8] is has been shown that $\langle\psi|\rho|\psi\rangle \leq \langle\psi|\sigma|\psi\rangle$ for all states ρ that can be reached from σ with *any probability* $p > 0$ (that is, $\sigma \to \rho$ under LOCC does not necessarily hold), implying that for this class of states the proportion of $|\psi\rangle\langle\psi|$ cannot even be increased with non-trace-preserving operations [14]. This is also true for ELOCC operations.

Let $\sigma \in S(\mathcal{H})$ be such a state, and let $\omega \in S(\tilde{\mathcal{H}}) = S(\tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B)$ be an appropriate catalyst state. The above transformation then amounts to a map

$$\begin{aligned} \sigma \otimes \omega &\mapsto \rho \otimes \omega \\ &= \frac{\sum_i (A_i \otimes B_i)(\sigma \otimes \omega)(A_i \otimes B_i)^{\dagger}}{\mathrm{tr}[\sum_i (A_i \otimes B_i)(\sigma \otimes \omega)(A_i \otimes B_i)^{\dagger}]}, \end{aligned} \tag{19}$$

where A_i and B_i satisfying $\sum_i A_i^{\dagger} A_i \leq 1$ and $\sum_i B_i^{\dagger} B_i \leq 1$ act only in $\mathcal{H}_A \otimes \tilde{\mathcal{H}}_A$ and $\mathcal{H}_B \otimes \tilde{\mathcal{H}}_B$, respectively. The quantity of interest is now the fidelity $F = \langle\psi|\rho|\psi\rangle$ of ρ with respect to $|\psi\rangle\langle\psi|$. It is given by

$$\begin{aligned} F(\lambda) = \mathrm{tr}_{\tilde{\mathcal{H}}} \sum_i \{&\lambda\langle\psi|[(A_i \otimes B_i)(|\psi\rangle\langle\psi| \otimes \omega)(A_i \otimes B_i)^{\dagger}]|\psi\rangle \\ &+ (1 - \lambda)\langle\psi|[(A_i \otimes B_i)(\zeta \otimes \omega)(A_i \otimes B_i)^{\dagger}]|\psi\rangle\}/\mathcal{N}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{N} = \sum_i \mathrm{tr}[&(A_i \otimes B_i)\{[\lambda|\psi\rangle\langle\psi| + (1 - \lambda)\zeta] \otimes \omega\} \\ &\times (A_i \otimes B_i)^{\dagger}]. \end{aligned}$$

$dF^2(\lambda)/d^2\lambda = C/\mathcal{N}^3$ with a number C independent of λ, and one can argue, as in the case of local operations without a catalyst state, the following [8]: The sign of the second derivative of the function $f(\lambda) = F(\lambda) - \lambda$ is constant for all $\lambda \in (0,1)$, and, therefore, this function is convex, concave, or linear. At $\lambda = 0$, $f(0) \geq 0$, as $f(\lambda) \geq -\lambda$ for $\lambda \in (0,1)$, and $f(1) \leq 0$. $f(\lambda_0) \leq 0$ follows from the fact that the map equation (20) cannot transform the state pertaining to λ_0 to an entangled state. Hence, $f(\lambda) \leq 0$ for all $\lambda \in [\lambda_0, 1)$, i.e., the proportion of $|\psi\rangle\langle\psi|$ can only decrease.

Small transformations and catalysis for pure and mixed states.—So far, the findings in the pure state case and those for mixed states have suggested rather similar behavior of both sets of states with respect to LOCC and ELOCC operations. However, things are quite different in the next issue concerning the possibility to enhance the range of accessible states with catalyst states in "small" transformations.

Lemma 2.— For all pure states $|\psi\rangle \in \mathcal{H}$ and all pure catalyst states $|\tilde{\psi}\rangle \in \tilde{\mathcal{H}}$ there exists a $\delta > 0$, such that

$$|\psi\rangle \not\rightarrow |\phi\rangle \text{ under LOCC} \Rightarrow |\psi\rangle \not\rightarrow |\phi\rangle \text{ under ELOCC}$$

for all $|\phi\rangle \in \mathcal{H}$ with $|\langle\psi \mid \phi\rangle|^2 > 1 - \delta$.

Proof: Let $\alpha_1, \dots, \alpha_N$ be the ordered lists of eigenvalues of $\mathrm{tr}_A[|\psi\rangle\langle\psi|]$, $N = \dim[\mathcal{H}_A]$, and let $\gamma_1, \dots, \gamma_M$ be the corresponding list of the pure catalyst state, $M = \dim[\tilde{\mathcal{H}}_A]$. Now let $\varepsilon > 0$ and call an ε list a list $\beta_1, \dots, \beta_N$ with $1 \geq \beta_1 \geq \ldots \geq \beta_N \geq 0$ that has the property $|\beta_i - \alpha_i| < \varepsilon$ for all $i = 1, \dots, N$. There exists an $\varepsilon > 0$ such that, for all ε lists $\beta_1, \dots, \beta_N$, the statement that $\alpha_i\gamma_j > \alpha_k\gamma_l$ for some $i, k \in \{1, \dots, N\}, j, l \in \{1, \dots, M\}$ implies that $\beta_i\gamma_j > \beta_k\gamma_l$. This ε is, in the following, referred to as $\tilde{\varepsilon}$. Moreover, there exists a $\delta > 0$ such that, for each $|\phi\rangle \in \mathcal{H}$ with $|\langle\psi \mid \phi\rangle|^2 > 1 - \delta$, the ordered eigenvalues of $\mathrm{tr}_A[|\phi\rangle\langle\phi|]$ form a $\tilde{\varepsilon}$ list (and, hence, for such states it is not possible that $\beta_i\gamma_j < \beta_k\gamma_l$ and $\alpha_i\gamma_j > \alpha_k\gamma_l$). It follows that for all such $|\phi\rangle \in \mathcal{H}$ with $|\langle\psi \mid \phi\rangle|^2 > 1 - \delta$ the majorization relation $\mathrm{tr}_A[|\psi\rangle\langle\psi| \otimes |\tilde{\psi}\rangle\langle\tilde{\psi}|] \nprec \mathrm{tr}_A[|\phi\rangle\langle\phi| \otimes |\tilde{\psi}\rangle\langle\tilde{\psi}|]$ holds if $\mathrm{tr}_A[|\psi\rangle\langle\psi|] \nprec \mathrm{tr}_A[|\phi\rangle\langle\phi|]$. Finally, this implies the statement of Lemma 2. ☐

This is not true for mixed states, when the fidelity of two states σ and ρ is taken to be $F(\sigma, \rho) = \{\mathrm{tr}[(\sqrt{\sigma}\,\rho\sqrt{\sigma})^{1/2}]\}^2$ [15]. Indeed, there are states $\sigma \in S(\mathcal{H})$ such that for every $\delta > 0$ there are states $\rho \in S(\mathcal{H})$ with the property that $F(\sigma, \rho) > 1 - \delta$ and $\sigma \not\rightarrow \rho$ under LOCC, but $\sigma \rightarrow \rho$ under ELOCC. Such states can, e.g., be constructed using the class of states defined in Eqs. (8a), (9a), and (9b). For any given $\delta > 0$ there is a sufficiently small $\lambda > 0$ such that the fidelity satisfies $F(\sigma, \rho) > 1 - \delta$.

Hence, quite surprisingly, in the case of entanglement manipulations from an initial pure state to a close pure state, entanglement-assisted operations do not add any power to LOCC operations. To put it in different words, there is no catalysis for sufficiently close pure states. Yet, for mixed states there can be catalysis for such close states.

In this paper we have investigated the power of entanglement-assisted manipulation of entangled quantum systems in mixed states. Interestingly, the counterintuitive class of ELOCC operations has proven to be superior to mere LOCC operations also in the interior of the state space, for which such strong tools as the majorization criterion are not available. Yet, although these findings might contribute to the quest for a better understanding of mixed-state entanglement, there are numerous open problems. Stronger criteria for the possibility of certain entanglement transformation are urgently needed. Finally, it is our hope that this work will help to explore practical applications [16] of the strange phenomenon of catalysis.

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