Triangle Map: A Model of Quantum Chaos

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We study an area preserving parabolic map which emerges from the Poincaré map of a billiard particle inside an elongated triangle. We provide numerical evidence that the motion is ergodic and mixing. Moreover, when considered on the cylinder, the motion appears to follow a Gaussian diffusive process.

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The investigation of the quantum manifestations of classical dynamical chaos has greatly improved our understanding of the properties of quantum motion. Even though, besides some very special cases, the nonlinear terms prevent exact solution of the Schrödinger equation, still important useful information can be obtained concerning statistical properties of eigenvalues and eigenfunctions. An important discovery has been the phenomenon of quantum dynamical localization [1] which consists of the quantum suppression of deterministic classical diffusive behavior. This suppression takes place after a relaxation time scale t_R which is defined as the density ρ of the operative eigenstates [2], namely, those states which enter the initial conditions and therefore determine the dynamics. One finds $t_R \sim 1/\hbar^2$ in systems with normal diffusion in classical case, measured in natural units of time like the number of bounces in billiards. For $t < t_R$, the quantum motion mimics the classical diffusive behavior and relaxation to statistical equilibrium takes place. The remarkable fact is that quantum "chaotic" motion is dynamically stable as it was illustrated in [3]. This means that, unlike the exponentially unstable classical chaotic motion, in the quantum case errors in the initial conditions propagate only linearly in time. More precisely, besides the relaxation time scale t_R, a second very important time scale exists, the so-called random time scale $t_r \sim \ln\hbar$, below which also the quantum motion is exponential unstable. However, as noted in [2] $t_r \ll t_R$, and therefore the quantum diffusion and relaxation process takes place in the absence of exponential instability. It should be noticed that, even though the time scale t_r is very short, it diverges to infinity as $\hbar$ goes to zero and this ensures the transition to classical motion as required by the correspondence principle.

Therefore, typical quantum systems exhibit a new type of relaxation for which we do not have yet a physical description. In terms of the classical ergodic hierarchy, quantum systems can be at most mixing. While exponential instability is sufficient for a meaningful statistical description, it is not known whether or not it is also necessary. Several questions remain unanswered; e.g., there is no general relation between the rate of exponential instability and the decay of correlations. Moreover, as shown in [4], quantum systems provide examples which show that linear dynamical instability is not incompatible with exponential decay of Poincaré recurrences.

In a recent paper a physical example has been found [5], a billiard in a triangle, which has zero Kolmogorov-Sinai entropy (the instability is linear only in time) but which possesses the mixing property [6]. This characteristic makes systems of this type good candidates for the discussion of the above mentioned problems. In the present paper, starting from the discrete bounce map for the billiard in a triangle, we derive an area preserving, parabolic, classical map. In other words, the map is marginally stable; i.e., initially close orbits separate linearly with time. We will show that this map is mixing, with power law decay of correlations and exponential decay of Poincaré recurrences, and has a peculiar property: absence of periodic orbits. Moreover, when the map is considered on the cylinder, it exhibits normal diffusion with the corresponding Gaussian probability distribution.

Let us consider the following discontinuous skew translation on the torus, with symmetric coordinates $(x, y) \in \mathbb{T}^2 = [-1, 1) \times [-1, 1)$,

$$\begin{aligned} y_{n+1} &= y_n + \alpha\,\mathrm{sgn} x_n + \beta \qquad (\mathrm{mod} 2)\,, \\ x_{n+1} &= x_n + y_{n+1} \qquad (\mathrm{mod} 2)\,, \end{aligned} \tag{1}$$

where $\mathrm{sgn} x = \pm 1$ is the sign of x. The map (1), which we will call "the triangle map," is a parabolic, piecewise linear, one-to-one (area preserving) map, $\det J = 1$, $\mathrm{tr} J = 2$ with $J := \partial(y_{n+1}, x_{n+1})/\partial(y_n, x_n) \equiv 1$. It is known that (continuous) irrational skew translations [the above map (1) with $\alpha = 0$ and irrational β] are *uniquely ergodic* [7] and never mixing [8]; in fact, they are equivalent to interval exchange transformations. However, the triangle map may have more complicated dynamics, and we show below that discontinuity may provide a mechanism to establish the mixing property. Noninvertible piecewise linear 2D parabolic maps have been studied in Ref. [9].

The triangle map is related to the Poincaré map of the billiard inside the triangle with one angle being very small. Indeed, let us assume that the small angle of the billiard can

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be written as $\gamma = \pi/M$ with some integer $M \gg 1$. Then the billiard dynamics may be *unfolded* by means of reflections over the two long sides of the triangle into the dynamics inside a nearly circular $2M$-sided polygon. Within relative accuracy of $1/M$, the approximate Poincaré map inside such a polygon, relating two successive collisions with the short sides of the triangle—the outer boundary of the polygon—reads

$$\begin{aligned} v_{n+1} &= v_n + 2(u_n - [u_n] - \mu(-1)^{[u_n]}), \\ u_{n+1} &= u_n - 2v_{n+1}, \end{aligned} \tag{2}$$

where γu_n is the polar angle and γv_n is the angle of incidence of the trajectory in the nth collision. The symbol $[x]$ is the nearest integer to x. The parameter μ controls the asymmetry between the other two angles η, ζ of the triangle, namely, $\eta, \zeta = \pi/2 - \gamma(\frac{1}{2} \pm \mu)$, and we assume that the triangle has all angles smaller than $\pi/2$, i.e., $|\mu| \le \frac{1}{2}$. As shown in [5], the system is equivalent to the mechanical problem of three elastic point masses on a ring (here one particle being much lighter than the other two). It is interesting to note that in the scaled variables (u, v) the small parameter γ scales out from the map and the limit $\gamma \to 0$ simply means that the range of variables $u_n \in [0, 2\pi/\gamma]$, $v_n \in [-\pi/(2\gamma), \pi/(2\gamma)]$ becomes the entire plane $\mathbb{R}^2$. The above map can be compactified onto a torus $\mathbb{T}^2$ by considering one "primitive cell" $[u_n \pmod 2, v_n \pmod 1]$. After transforming the coordinates as $y_n = 2(-1)^n(u_n + v_n - \frac{1}{2}) \pmod 2$, $x_n = (-1)^n(u_n - \frac{1}{2}) \pmod 2$, we obtain the discontinuous skew translation of a torus (1) with $\alpha = 4\mu$ and $\beta = 0$.

In the following we consider the general case of the triangle map with parameters α and β being two independent irrationals. The particular case $\beta = 0$ will be briefly discussed at the end of the paper. We fix the parameter values $\alpha = (\frac{1}{2}(\sqrt{5} - 1) - e^{-1})/2$, $\beta = (\frac{1}{2}(\sqrt{5} - 1) + e^{-1})/2$, although qualitatively identical results were obtained for other irrational parameter values.

As a first step we make a detailed and careful test of ergodicity of the triangle map. To this end, following [10], we discretize the phase space $\mathbb{T}^2$ in a mesh of $N = N_1 \times N_1$ cells and then measure the number of cells $n(t)$ visited by a given orbit up to discrete time t. Computing the phase space averages $\langle\,\rangle$ by averaging over many randomized initial conditions we compare the quantity $r(t) = \langle n(t)/N \rangle$ thus obtained with the corresponding $r_{\mathrm{RM}}(t)$ for the *random model* in which each throw onto a mesh of N cells is completely random. As it is known, in the latter case, $r_{\mathrm{RM}}(t) = 1 - \exp(-t/N)$. The result shown in Fig. 1 provides strong evidence of (fast) ergodicity (without any secondary time scales): namely, the exploration rate $r(t)$ of phase space for the triangle map approaches 1 as $t \to \infty$ and, for sufficiently fine mesh N, is arbitrarily close to the random model $r_{\mathrm{RM}}(t)$.

Having established with reasonable confidence that the triangle map is ergodic, we now turn our attention to the *mixing property*. This amounts to showing asymptotic decay of time-correlation functions of arbitrary L^2 observables. The extensive numerical experiments we have performed suggest that arbitrary time-correlation functions decay asymptotically with a power law $\langle f(t)g(0)\rangle \propto t^{-\sigma}$ with the value of the exponent σ close to $\sigma = 3/2$. In Fig. 2a we show the decay of autocorrelations of a typical observable $f = \cos(\pi y)$ [11]. The property of mixing and the nature of decay of correlations are intimately related to the spectral properties of the *unitary* evolution (Koopman) operator over L^2 space of observables over $\mathbb{T}^2$. The value $\sigma > 1$ we have empirically found

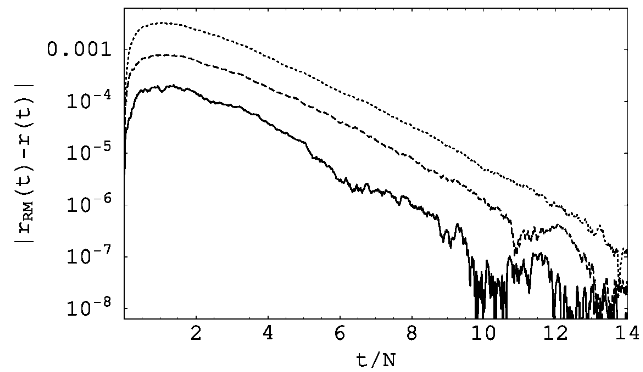

FIG. 1. Deviation of the filling rate from the random model in log-normal scale for three different mesh sizes $N = 10^4$ (dotted), $N = 10^5$ (dashed), and $N = 10^6$ (solid curve).

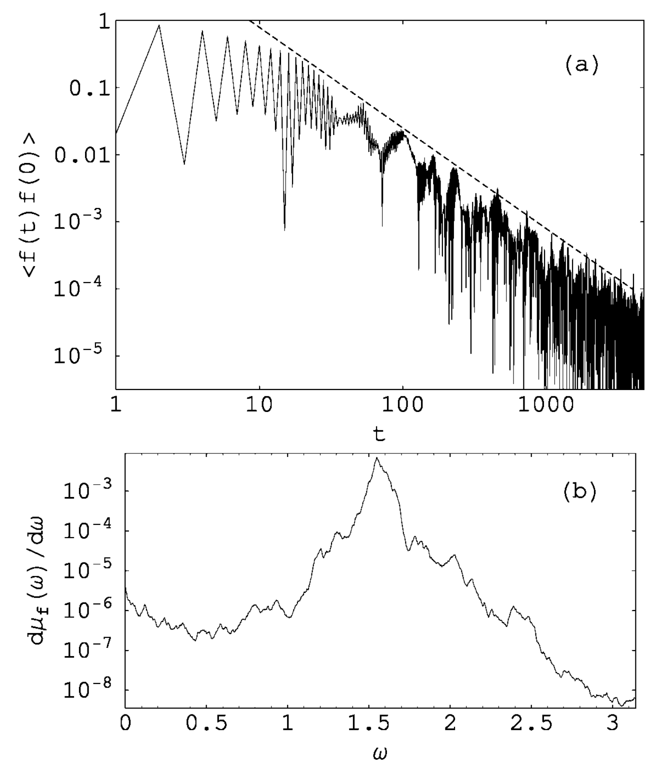

FIG. 2. The autocorrelator (a) $C(t) = \langle\cos(\pi y_t)\cos(\pi y_0)\rangle$ averaged over 2×10^6 orbits of length 16 384 with randomized initial conditions. The dashed line has slope $-3/2$. In (b) we show the corresponding spectral density. Note that peak at $\omega = \pi/2$ indicates a strong component of period 4.

implies an absolutely continuous spectrum. Performing the inverse Fourier transform of the time autocorrelator $C(t) := \langle f(t)f(0)\rangle = \int d\mu_f(\omega)e^{i\omega t}$ one calculates the spectral density $d\mu_f(\omega)/d\omega$ which should be a *nonsingular and continuous* but *nonsmooth and nonanalytic* function, according to the (power law, $\sigma > 1$) nature of decay of correlations. In Fig. 2b we show the spectral density $d\mu_f(\omega)/d\omega$ which is apparently continuous but not a continuously differentiable function. In fact, we suggest that the discontinuities of the derivative are *dense* in order to ensure the correlation decay with the power σ which is between 1 and 2.

A very efficient tool for investigating the statistical properties of dynamical systems is the study of Poincaré recurrences, i.e., the probability $P(t)$ for an orbit to stay outside a specific subset $\mathcal{A} \subset \mathbb{T}^2$ for a time longer than t. In Fig. 3 we plot the Poincaré recurrence probability $P(t)$ for the map (1) and for several different subsets of the form $\mathcal{A} = [0,b] \times [0,b]$. The result is quite unexpected. Indeed, for any sufficiently small set (small b) the return probability appears to decay *exponentially* $P(t) \propto \exp(-\lambda t)$. Moreover, the exponent λ is very close to the Lebesgue measure of the subset $\mu = |\mathcal{A}|$, as in the case of the random model of completely stochastic dynamics for which $P_{\mathrm{RM}}(t) = \exp(-\mu t)$. Therefore the triangle map, which is characterized by a linear separation of orbits, exhibits exponential decay of Poincaré recurrences, typical of hyperbolic systems. Notice that in strongly chaotic systems with positive Lyapunov exponents, the presence of a zero measure of marginally unstable orbits (e.g., bouncing balls in the Sinai billiard) leads to a power law decay of Poincaré recurrences. The simultaneous presence in our model of a power law decay of correlations and exponential decay of Poincaré recurrences is a fact for which, so far, we have no explanations. Indeed, even if there are no general rigorous theorems, it has been conjectured that correlations of dynamical observables have the same decay as the integrated Poincaré recurrences [12], namely, $C(t) \sim P_i(t) := \int_t^\infty d\tau\, P(\tau)$ for asymptotically long times t. This relation is obviously violated in our model (1), and this interesting point requires further investigations.

Our last step is the investigation of the diffusive properties of the system. To this end we consider the triangle map on the cylinder [$y \in (-\infty,\infty)$]. In order to take into account the constant drift of y_n with "velocity" β we find it convenient to introduce a new integer variable $p_n \in \mathbb{Z}$:

$$y_n = y_0 + \beta n + \alpha p_n, \qquad (3)$$

which has, by definition, vanishing initial value $p_0 = 0$, and then study the diffusive properties in the variable p_n. Our numerical results shown in (Fig. 4) provide clear numerical evidence for normal diffusive behavior. In particular, we obtained a very accurate linear increase of the second moment (notice the long integration time),

$$\langle (p_{n+t} - p_n)^2\rangle = \langle p_t^2\rangle = Dt, \qquad (4)$$

with diffusion coefficient $D \approx 1.654$. The almost perfect Gaussian distributions of $p_t - p_0 = p_t$ obtained at

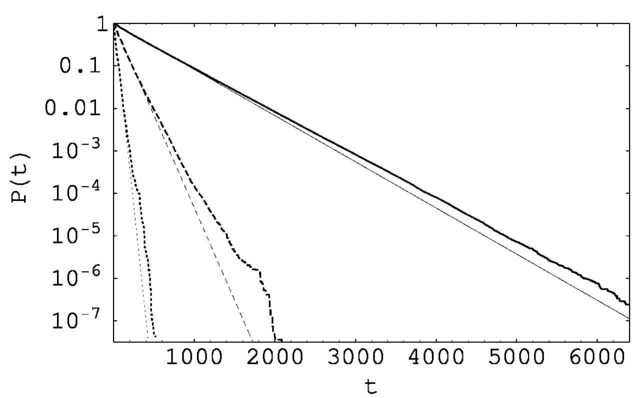

FIG. 3. The Poincaré recurrence probabilities $P(t)$ for three different subsets $[0, 0.1] \times [0, 0.1]$ (solid), $[0, 0.2] \times [0, 0.2]$ (dashed), and $[0, 0.4] \times [0, 0.4]$ (dotted curve). *Thick* curves give numerical data obtained by computing the return probability to the subset $\mathcal{A}$ for a single orbit of length 3×10^{11}. *Thin* curves are theoretical estimates for fully random dynamics, $P_i(t) = \exp(-\mu t)$, where μ is the relative Lebesgue measures of the above sets, namely, $\mu = 1/400, 1/100, 1/25$, respectively.

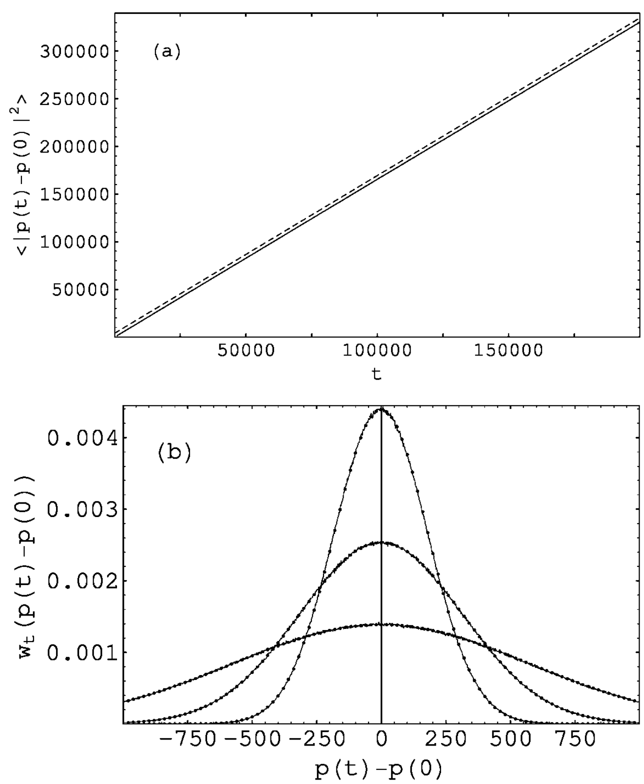

FIG. 4. The normal diffusion of the map (1) on a cylinder. In (a) we show averaged squared displacement of an average over 10^7 orbits of length 2×10^5 (solid line) compared with the straight line (dashed) with slope $D = 1.654$. In (b) we show the corresponding distribution of displacements at three different times ($t = 20\,000, 60\,000, 200\,000$, solid curves) which are in perfect agreement with the solutions of the diffusion equation (Gaussians with variance Dt, dotted curves).

different times (Fig. 4b) indicate that we are in the presence of a normal Gaussian process. Note that dynamics (1) can be rewritten in terms of a closed map on an integer lattice $\mathbb{Z}^2$ with an explicit "time dependence," namely, rewrite also the variable x_n in terms of the integer variable $q_n \in \mathbb{Z}$,

$$x_n = x_0 + y_0 n + \beta \frac{n(n+1)}{2} + \alpha q_n \qquad (\text{mod}2)\,, \tag{5}$$

and the map (1) becomes equivalent to an integer system

$$\begin{aligned} p_{n+1} &= p_n + (-1)^{[x_0 + y_0 n + \beta n(n+1)/2 + \alpha q_n - (1/2)]}, \\ q_{n+1} &= q_n + p_{n+1}\,, \end{aligned} \tag{6}$$

with fixed initial conditions $p_0 = q_0 = 0$. Here the original initial conditions x_0, y_0 enter as parameters.

We note the trivial but important fact that the triangle map possesses *no periodic orbits* when the parameter β is *irrational* and α and β are *incommensurable.* Therefore, the general argument of Ref. [5] using parabolic periodic orbits cannot be used to derive the $1/t^2$ decay of Poincaré recurrence probabilities. It has been verified numerically that the nonexistence of periodic orbits is indeed responsible for exponential decay of Poincaré recurrence probability: When we replaced irrational β with a crude rational approximation we obtained a very clean crossover from initial exponential decay $\exp(-\mu t)$ to an asymptotic power law $P(t) \propto 1/t^2$ due to the existence of (long) periodic orbits. Our map thus provides quite a pathological example from the point of view of semiclassical periodic orbit theory, hence we pose an interesting question: which classical structure underpins the spectral fluctuations of the quantization of the triangle map (1)? (See also Ref. [13] for skew translations, $\alpha = 0$.)

An interesting special case of the triangle map is $\beta = 0$ which, as discussed above, describes the dynamics of an elongated triangle (2). Here two cases should be distinguished: (i) The parameter α $(= 4\mu)$ is *rational* $\alpha = 2k/l$, with $k, l \in \mathbb{Z}$, then the dynamics is *pseudo-integrable* and confined onto l-"valued" invariant curves $(y_n - y_0)l$ (mod2) = 0. (ii) The parameter α is *irrational,* then the dynamics has been found to be *ergodic.* However, ergodic properties turn out to be *weak* (see also [14]) and the rate of ergodicity is very slow as opposed to the general case $\beta \neq 0$: It has been shown that the number of different values of coordinate y_n taken by a single orbit up to the discrete time T, $0 \leq n < T$, grows extremely slowly, as $\propto \ln T$. A similar property has been found for triangular billiards in which one angle is rationally related with π, e.g., right triangles [5,15]. In addition, numerically computed correlation functions of (1) with $\beta = 0$, such as $\langle \cos(\pi y_t) \cos(\pi y_0) \rangle$, show perhaps a tendency to decay as power laws but with a small exponent σ around 0.1. It is fair to say that, in this case, it is difficult to judge definitively, based on numerical experiments, on the property of mixing even though it cannot be excluded.

In this paper we have shown that a Gaussian diffusive process and mixing behavior can take place in a simple area preserving map without dynamical exponential instability. One may argue that parabolic maps are nongeneric and therefore irrelevant for the description of physical systems. However, the results presented here show that a meaningful statistical description is possible without the strong property of exponential instability. Even if the model discussed here is nongeneric in the context of classical systems, it can describe the typical mechanism of quantum relaxation. Therefore it can play an important role in understanding and describing the quantum chaotic motion in analogy to the one played by the Arnold cat map for classical systems.

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[1] G. Casati, B. V. Chirikov, J. Ford, and F. M. Izrailev, in *Lecture Notes in Physics* (Springer-Verlag, Berlin, 1979), Vol. 93, p. 334.

[2] G. Casati and B. V. Chirikov, *Quantum Chaos* (Cambridge University Press, Cambridge, 1995), p. 14.

[3] G. Casati *et al.,* Phys. Rev. Lett. **56**, 2437 (1986).

[4] G. Casati, G. Maspero, and D. L. Shepelyansky, Phys. Rev. E **56**, R6233 (1997).

[5] G. Casati and T. Prosen, Phys. Rev. Lett. **83**, 4729 (1999).

[6] However, from a rigorous point of view, mixing property of (generic) polygonal billiards has not (yet) been established. For a review of mathematical results on polygonal billiards see E. Gutkin, J. Stat. Phys. **83**, 7 (1996).

[7] H. Furstenberg, Am. J. Math. **83**, 573 (1961).

[8] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, New York, 1982).

[9] K. Życzkowski and T. Nishikawa, Phys. Lett. A **259**, 377 (1999); P. Ashwin *et al.,* chao-dyn/9908017.

[10] M. Robnik *et al.,* J. Phys. A **30**, L803 (1997).

[11] We note that the time range for the plot of Fig. 2a has been chosen by comparing the numerical data with the statistical error, estimated as $\approx 5 \times 10^{-5}$ at the maximal time. Because of the law of large numbers the statistical error of $C(t)$ for fixed t decays as $\sim 1/\sqrt{NT}$ where N and T are the number and length of orbits, respectively. Therefore, increasing the integration time by a factor of 10 would require $10^{1.5}$ smaller statistical error, i.e., 10^3 times more CPU time. Since the data of Fig. 2 required 2 weeks of CPU time on an up-to-date workstation, results on asymptotic behavior of $C(t)$ of Fig. 2 cannot be significantly improved at present.

[12] R. Artuso, Physica (Amsterdam) **131D**, 68 (1999).

[13] A. Bäcker and G. Haag, J. Phys. A **32**, L393 (1999).

[14] L. Kaplan and E. J. Heller, Physica (Amsterdam) **121D**, 1 (1998).

[15] R. Artuso, G. Casati, and I. Guarneri, Phys. Rev. E **55**, 6384 (1997).