Meissner-London Currents in Superconductors with Rectangular Cross Section

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Exact analytic solutions are presented for the magnetic moment and screening currents in the Meissner state of superconductor strips with rectangular cross section in a perpendicular magnetic field and/or with transport current. The extension to finite London penetration is achieved by an elegant numerical method which works also for disks. The surface current in the specimen corners diverges as $l^{-1/3}$ where l is the distance from the corner. This enhancement reduces the barrier for vortex penetration and should increase the nonlinear Meissner effect in d-wave superconductors.

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The main feature of superconductors is that they expel weak magnetic fields H from their interior. This Meissner effect was described quantitatively by the London brothers, who showed that H penetrates exponentially to the London depth λ [1], and by Ginzburg and Landau and by Pippard, who introduced the superconducting coherence length ξ. In extreme type-II superconductors with $\lambda \gg \xi$, the correction caused by finite ξ to the Meissner-London state usually may be disregarded, but now the penetration of magnetic flux in the form of Abrikosov vortices has to be considered. Vortex penetration is governed by surface barriers which increase with decreasing ξ. Both the microscopic Bean-Livingston barrier [2,3] and the macroscopic geometric barrier [4–7] depend on the surface screening currents flowing in the Meissner-London state. These currents crucially depend on the specimen shape and on the orientation of the applied magnetic field H. The screening current is particularly large near sharp edges, where it causes a reduction of the field of first penetration of vortices and possibly increases the nonlinear Meissner effect in d-wave superconductors [8,9]. Knowledge of the Meissner-London state of thin platelets is also required for a correct evaluation of certain precision measurements of the London penetration depth λ [9].

In spite of its fundamental nature, exact solutions of London theory exist only for the trivial (and less important for experiments) longitudinal geometry of infinite slabs and cylinders in parallel H, and for the sphere and infinite cylinder in arbitrarily oriented H [1]. Even for the ideal Meissner state, i.e., the limit $\lambda \to 0$, the only nontrivial solution we know of is the ellipsoid [10,11] and strips with elliptic [12] and oval [13] cross sections. In general, when λ is much smaller than the smallest extension of the superconductor, the surface screening current is $J = |H_\parallel|$ where $H_\parallel$ is the component of $\mathbf{H}(\mathbf{r})$ at and parallel to the surface, and its density is approximately

$$j = (J/\lambda)\exp(-\delta/\lambda), \qquad (1)$$

where δ is the distance from the surface.

In the present paper we first give an exact solution for the surface current and magnetic moment in the ideal Meissner state ($\lambda = 0$) of an infinitely long strip with rectangular cross section in a perpendicular magnetic field H or with transport current I. We then show how the Meissner-London state of this strip (and of circular disks) can be computed for arbitrary λ. In all these cases the solutions for $H > 0$, $I = 0$ and $H = 0$, $I > 0$ may be superimposed linearly to give the general (less symmetric) result for simultaneously applied H and I. The rectangular cross section of the strip fills the area $-a \le x \le a$, $-b \le y \le b$; the currents flow along the strip $\|z$, and H is applied along y; see inset in Fig. 1. The result for arbitrary orientation of H is obtained by linear superposition of the solutions with $H\|y$ and $H\|x$.
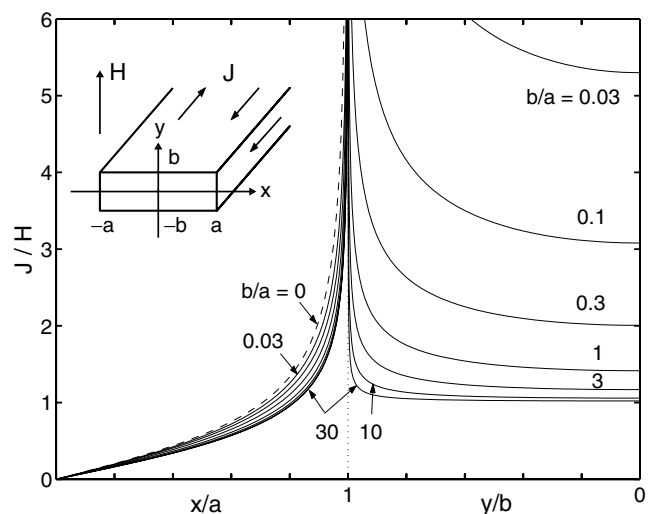

FIG. 1. The Meissner surface currents $J(x, b)$ and $J(a, y)$, Eqs. (4) and (5), in strips with various aspect ratios b/a; see inset. The dashed line gives the thin strip limit, Eq. (12).

The surface currents of an ideally diamagnetic rectangular strip in a perpendicular field $H \| y$ or with transport current $I \| z$ can be calculated by conformal mapping of the half plane on a rectangle. We give here the main results starting with the case $H > 0$, $I = 0$; details will be published elsewhere. Defining the universal function

$$f(s, m) = ms \int_0^1 \frac{\sqrt{1 - s^2 t^2}}{\sqrt{1 - ms^2 t^2}} \, dt, \qquad (2)$$

[f is the sum of two incomplete elliptic integrals, $f(s, m) = E(\theta, k) - (1 - k^2)F(\theta, k)$, $s = \sin\theta$, $m = k^2$, $0 \le s \le 1$, $0 \le m \le 1$ (see textbooks)] we may write the screening currents and the magnetic moment *in parametric form,* with curve parameters s and m. First we find $m(b/a)$ from

$$\frac{b}{a} = \frac{f(1, m)}{f(1, 1 - m)} \qquad (3)$$

and then the currents as functions of $x(s)$ and $y(s)$:

$$\frac{J(x, b)}{H} = \frac{s}{\sqrt{1 - s^2}}, \qquad \frac{x}{a} = \frac{f(s, 1 - m)}{f(1, 1 - m)}, \qquad (4)$$

$$\frac{J(a, y)}{H} = \frac{\sqrt{1 - ms^2}}{\sqrt{m(1 - s^2)}}, \qquad \frac{y}{b} = \frac{f(s, m)}{f(1, m)}. \qquad (5)$$

The (negative) magnetic moment of the strip per unit length along z is

$$\frac{-M}{\pi a^2 H} = \frac{1 - m}{[f(1, 1 - m)]^2}, \qquad (6)$$

with $m = m(b/a)$ from Eq. (3). These exact results are shown in Figs. 1 and 2.

Some useful approximations and limiting cases are as follows. For all aspect ratios $0 < b/a < \infty$ one has

$$m = \frac{1}{2} + \frac{1}{2} \tanh\left(0.463 \ln\frac{b}{a}\right) + \epsilon, \qquad (7)$$
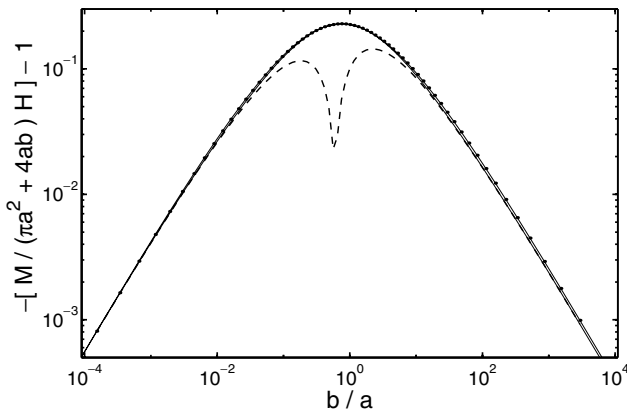

FIG. 2.　Magnetic moment of Meissner strips versus the aspect ratio b/a, Eq. (6) (solid line). The solid line with dots shows the fit (8), and the dashed line the limits (9) and (10).

with $|\epsilon| \le 0.0018$. The exact limits are $m \to 4b/\pi a$ ($b \ll a$) and $m \to 1 - 4a/\pi b$ ($b \gg a$). The magnetic moment (6) for all ratios b/a is well fitted by

$$\frac{-M}{(\pi a^2 + 4ab)H} = 1 + \exp(1.6875 - \sqrt{p}) - \epsilon, \qquad (8)$$

with $p = 10 + |\ln(b/a) + 0.288|^{1.968}$ and deviation $0 \le \epsilon \le 0.004$; see Fig. 2. Formula (8) is more accurate than the fit given in Ref. [14]. The exact limits are (Fig. 2)

$$\frac{-M}{\pi a^2 H} = 1 + \frac{2b}{\pi a}\left(\ln\frac{4\pi a}{b} - 1\right), \qquad b \ll a, \qquad (9)$$

$$\frac{-M}{4abH} = 1 + \frac{a}{\pi b}\left(\ln\frac{4\pi b}{a} + \frac{1}{2}\right), \qquad b \gg a. \qquad (10)$$

The surface currents diverge at the corners symmetrically as $l^{-1/3}$ where l is the distance from the corner, e.g., $l = a - x$ and $l = b - y$. Near the corners one has

$$J_{\text{corner}} = H\left[\frac{(1 - m)}{3\sqrt{m}\,f(1, m)}\frac{b}{l}\right]^{1/3}. \qquad (11)$$

At the equator $J(a, 0) = H/\sqrt{m}$ holds and near the poles $J(x, b)/H \approx (x/a)f(1, 1 - m)/(1 - m)$ which equals $\pi x/4a$ at $b \gg a$ and x/a at $b \ll a$. For long slabs one has $J(a, y) \approx H$ except near the corners [see Eq. (11)]. For thin strips ($b \ll a$) Eq. (4) yields

$$J(x, b) = H x (a^2 - x^2)^{-1/2}, \qquad (12)$$

in agreement with the known sheet current $2J(x)$ [15,16]. On the edge of thin strips one has the fit

$$J(a, y) = Hm^{-1/2}[1 - (y/b)^2]^{-0.31} \qquad (13)$$

with relative error $<1\%$ for $y/b < 0.92$ or $<2\%$ for $y/b < 0.97$. More limiting expressions and fits are easily derived from the exact equations (2)–(6).

For the strip with current $I > 0$ and $H = 0$ we find the Meissner surface currents

$$J(x, b) = \frac{I}{2\pi a}\frac{f(1, 1 - m)}{\sqrt{1 - m}\sqrt{1 - s^2}},$$
$$\frac{x}{a} = \frac{f(s, 1 - m)}{f(1, 1 - m)}, \qquad (14)$$

$$J(a, y) = \frac{I}{2\pi b}\frac{f(1, m)}{\sqrt{m(1 - s^2)}}, \qquad \frac{y}{b} = \frac{f(s, m)}{f(1, m)}. \qquad (15)$$

These expressions are invariant when interchanging a, b and x, y, which replaces m by $1 - m$. In the corners the current again diverges as $l^{-1/3}$; cf. Eq. (11):

$$J_{\text{corner}} = \frac{I}{2\pi b}\left[\frac{f(1, m)^2}{3\sqrt{m - m^2}}\frac{b}{l}\right]^{1/3}. \qquad (16)$$

Accounting for a small but finite London depth $\lambda \ll a, b$, the screening currents penetrate exponentially from the surface, e.g., $j(x, y) = J(x, b)\lambda^{-1} \times \exp[(y - b)/\lambda]$ near $y = b$. The magnetic moment $M = \int dx \int dy \, x \, j(x, y)$ is thus slightly reduced to approximately

$$M(a, b, \lambda) \approx M(a - \lambda, b - \lambda, 0), \quad (17)$$

with $M(a, b, 0)$ from Eqs. (6),(8)–(10). Similar formulas are valid for specimens of any shape if the radius of curvature of the surface is much larger than λ. For our strip, however, the sharp rectangular corners give an additional contribution δM_{corner} to $M(\lambda) - M(0)$. When $[\lambda/\min(a, b)]^{1/3} \ll 1$, one has from Eq. (11)

$$\delta M_{\text{corner}} \propto (\lambda^2/ab)^{1/3} M. \quad (18)$$

This nonanalytical term dominates in $\partial M/\partial \lambda$ and may explain some experimental findings in Ref. [9]. In the opposite case, $\lambda \gg a, b$, the vector potential of the induced current density is negligible, thus $j \approx -Hx/\lambda^2$ and

$$M(a, b, \lambda) \approx -(4a^3b/3\lambda^2)H. \quad (19)$$

Next we show how the current density $j(x, y)$ and magnetic moment M of thick strips (and of disks) for finite London depth λ can be obtained in an elegant way, avoiding the calculation and cutoff [17] of the magnetic field around the strip. The static London equation reads

$$-\lambda^2 \mu_0 \mathbf{j} = \mathbf{A} = \mathbf{A_j} + \mathbf{A}_a, \quad (20)$$

where $\mathbf{A_j}$ is the vector potential of the supercurrent density j and $\mathbf{A}_a$ is the vector potential of the applied field, e.g., $\mathbf{A}_a = -\hat{\mathbf{z}}x\mu_0 H$ for strips ($\hat{\mathbf{z}}$ = unit vector along z). Inverting $\mu_0 \mathbf{j} = -\nabla^2 \mathbf{A_j}$ we get for thick strips

$$A_j(\mathbf{r}) = -\mu_0 \int d^2 r' \frac{\ln|\mathbf{r} - \mathbf{r}'|}{2\pi} j(\mathbf{r}'). \quad (21)$$

with $\mathbf{r} = (x, y)$, $\mathbf{A} = \hat{\mathbf{z}}A$, $\mathbf{j} = \hat{\mathbf{z}}j$. The integration is over the strip cross section. From Eqs. (20) and (21) we have

$$A_a(\mathbf{r}) = \mu_0 \int d^2 r' \left[\frac{\ln|r - r'|}{2\pi} - \lambda^2 \delta(\mathbf{r} - \mathbf{r}') \right] j(\mathbf{r}'), \quad (22)$$

Solving for j and using $A_a = -x\mu_0 H$ we obtain

$$j(\mathbf{r}) = H \int d^2 r' K(\mathbf{r}, \mathbf{r}') x', \quad (23)$$

$$K(\mathbf{r}, \mathbf{r}') = \left[-\frac{\ln|\mathbf{r} - \mathbf{r}'|}{2\pi} + \lambda^2 \delta(\mathbf{r} - \mathbf{r}') \right]^{-1}. \quad (24)$$

The λ dependent integral kernel $K(\mathbf{r}, \mathbf{r}')$ may be computed by choosing $\mathbf{r}$ and $\mathbf{r}'$ on a grid and inverting the resulting matrix similar to what is shown in Refs. [14,16].

From Eq. (23) the current density induced by H in the Meissner-London state is obtained by a simple integration over x and y. Because of the symmetry of $j(x, y) = j(x, -y) = -j(-x, y) = -j(-x, -y)$ it suffices to inte-
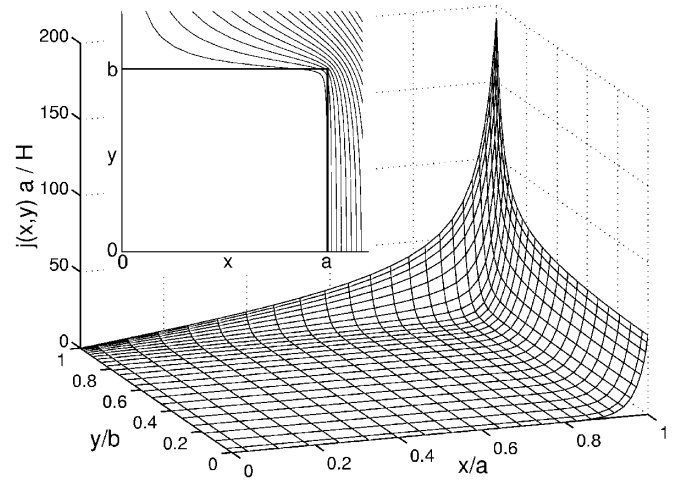

FIG. 3. The current density $j(x, y)$ along a London strip with square cross section ($a = b$) and London depth $\lambda/a = 0.025$ in a perpendicular magnetic field H. A quarter of the cross section is shown. Note the sharp but finite peak in the corner. The inset shows the magnetic field lines.

grate over a quarter of the strip cross section, $0 \le x \le a$, $0 \le y \le b$, if the kernel is made symmetric.

Similar equations follow for the strip with current I,

$$j(\mathbf{r}) = I \int d^2 r' K(\mathbf{r}, \mathbf{r}') \bigg/ \int d^2 r \int d^2 r' K(\mathbf{r}, \mathbf{r}'), \quad (25)$$

with the same kernel, Eq. (24), which, however, has a different symmetric form since now $j(x, y) = j(x, -y) = j(-x, y) = j(-x, -y)$ holds. For a strip with both applied field H and current I the two solutions, Eqs. (23) and (25) may be superimposed. The same method works for thick disks in axial field if the appropriate kernel K is used, similar to what is shown in Ref. [16].

From the current density $j(x, y)$ the magnetic induction $\mathbf{B}(x, y) = (B_x, B_y) = (\partial A/\partial y, -\partial A/\partial x)$ is obtained via
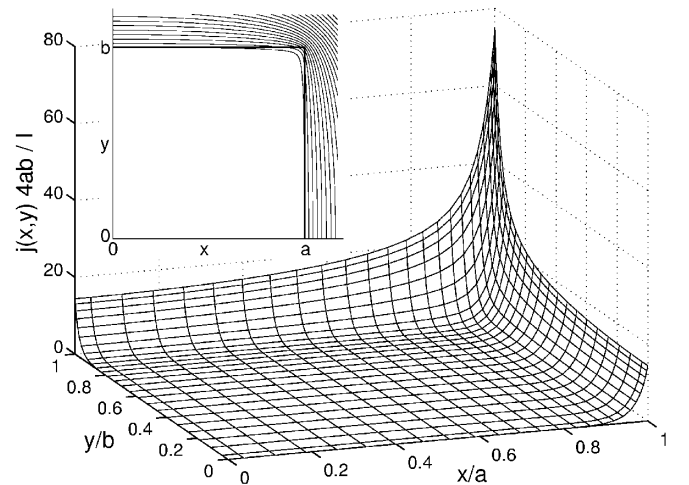

FIG. 4. The current density $j(x, y)$ along the London strip of Fig. 3 ($a = b = 40\lambda$) but with transport current I and no applied field, $H = 0$.
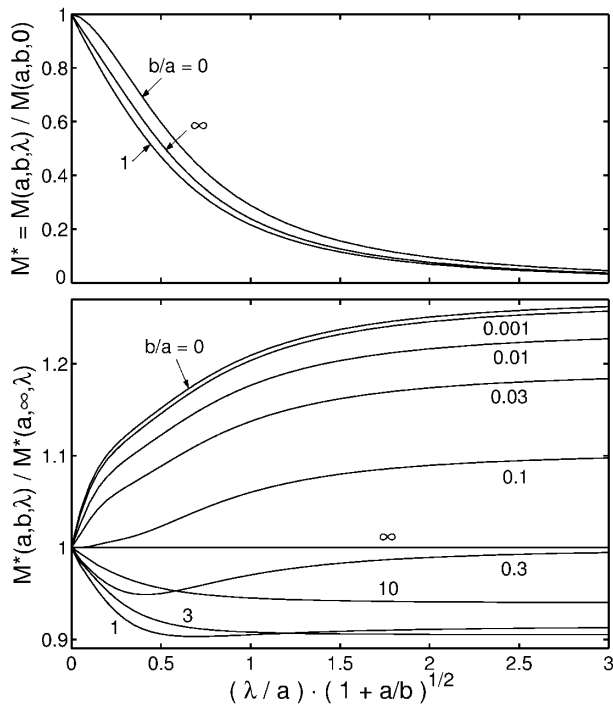
FIG. 5.   Reduced magnetic moment of London strips $M^* = M(a, b, \lambda)/M(a, b, 0)$ for various aspect ratios b/a. Top: Plotted versus a scaled London depth $(\lambda/a)\sqrt{1 + a/b}$ the curves $M^*(a, b, \lambda)$ almost collapse. Bottom: $M^*(a, b, \lambda)$ referred to the longitudinal limit $M^*(a, \infty, \lambda)$, Eq. (26).

the Biot-Savart law or by using Eq. (21) to get $A(x, y) = A_j + A_a$. In strip geometry the magnetic field lines are simply the contour lines of $A(x, y)$. In the London case inside the superconductor these field lines coincide with the contour lines of the current density since $A = -\mu_0 \lambda^2 j$; see Figs. 3 and 4.

Figures 3 and 4 show that the current density exhibits a sharp finite peak in the corners; for the depicted case $a = b = 40\lambda$ this enhancement is $j(a, b)/j(a, 0) = 3.7$ for $H > 0$, and $j(a, b)/j(a, 0) = 5.2$ for $I > 0$. For small λ, this enhancement increases as $\lambda^{-1/3}$. Clearly, this current peak favors the nucleation of vortex loops [2] at the corners and thus reduces the penetration field. It may also enhance the nonlinear Meissner effect [8,9].

Figure 5 shows the magnetic moment $M(a, b, \lambda)$ of London strips with various aspect ratios b/a as a function of λ and normalized to the ideal Meissner moment $M(a, b, 0)$, Eq. (6), and to the known $M(a, \infty, \lambda)$ of the infinite slab. One has

$$\lim_{b \to \infty} \frac{-M(a, b, \lambda)}{4abH} = 1 - \frac{\lambda}{a} \tanh \frac{a}{\lambda}, \qquad (26)$$

with the limiting cases $1 - \lambda/a$ for $\lambda \ll a$ and $a^2/(3\lambda^2)$ for $\lambda \gg a$. For thin strips with $b \ll a$ and $b < \lambda$ we find the limits (see also the Ohmic strip and disk in Ref. [18])

$$\frac{-M}{\pi a^2 H} = \begin{cases} 1 - \frac{2\lambda^2}{\pi ab} \ln(5.2 \frac{ab}{\lambda^2}), & \lambda^2 \ll ab, \\ \frac{4ab}{3\pi\lambda^2} - \frac{2a^2 b^2}{\pi^2 \lambda^4} + \dots, & \lambda^2 \gg ab. \end{cases} \qquad (27)$$

Note the nonanalytic λ dependence of M for small λ, which can be seen with the curve $b/a = 0$ in Fig. 5.

In summary, we found the exact analytical solution for the magnetic moment and surface screening currents of long strips with rectangular cross section in the ideal-screening Meissner state generated by a homogeneous magnetic field H and/or transport current I. Accounting for a finite London penetration depth λ, we present some explicit limiting expressions and numerical results which show a high and sharp but finite peak of the current density $j(x, y)$ along the four corners. This sharp peak favors the penetration of magnetic vortices from the corners, in the form of quarter loops spanning the corner. From the known $j(x, y)$ the exact shape and growth of these loops can be obtained in principle, and thus both the microscopic Bean-Livingston barrier [2,3] and the macroscopic geometric barrier [4–7], as well as the thermally activated penetration of vortex loops from the corners can be investigated in detail.

Finally, this pronounced current peak is expected to enhance the nonlinear Meissner effect predicted in d-wave superconductors [8] and to explain its dependence on the sharpness of the specimen corners as observed in Ref. [9].

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