Geometric Chaoticity Leads to Ordered Spectra for Randomly Interacting Fermions

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A rotationally invariant random interaction ensemble was realized in a single-j fermion model. A statistical approach reveals the random coupling of individual angular momenta as a source for the empirically known dominance of ground states with zero and maximum spin. The interpretation is supported by the structure of the ground state wave functions.

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The interplay of regular and chaotic features in many-body quantum dynamics is extensively studied both for simple models and for realistic applications to atomic, nuclear, and condensed matter physics, as well as for understanding properties of the QCD vacuum. Typical "shell-model" systems such as complex atoms [1] and nuclei [2] are described by the mean field and corresponding residual interaction. The density of the mean field configurations grows exponentially so that the interaction becomes effectively strong at sufficiently high excitation energy leading to generic chaotic features both in spectral statistics [3,4], and in properties of wave functions [1,2]. Studies of finite many-body systems have to account for the existence of constants of motion such as total angular momentum, isospin, and parity. Up to now, little attention was paid to the correlations between classes of states which are described by the same Hamiltonian but belong to different values of exact integrals of motion. An obvious and practically important example is angular momentum conservation in a finite Fermi system. For a sufficiently large dimension, the majority of states corresponds to a complicated quasirandom coupling of individual spins. This "geometric chaoticity" is used in evaluating the level density for a given J and plays an important role in the response to external fields, large amplitude collective motion, dissipation, and so on. The similarity of different J classes with respect to mixing was demonstrated [5,6] in the nuclear shell model by the studies of complexity, occupation numbers, strength functions and pairing properties. This raises also a question of existence of compound rotational bands [7] which would connect complicated states having different J but almost the same mixing.

A new angle of looking at the problem was introduced by Refs. [8,9] where the spectrum of a random but rotationally invariant Hamiltonian was obtained for a shell-model Fermi system. In spite of the random character of the two-body interaction, the fraction f_0 of the ensemble realizations with a ground state spin $J_0 = 0$ was much higher than the total statistical fraction f_0^s of $J = 0$ states in shell-model space. This result was confirmed in Refs. [10,11] as well as for the interacting boson model [12] being very robust and insensitive to the details of the ensemble. A new regular feature discovered in [10,12] was an excess of the probability $f_{J_{\max}}$ for the ground state to have the maximum possible spin $J_{\max}$. Below we show that the geometric chaoticity provides a base for explaining the main features of the pattern.

First we give a couple of trivial examples which point out the possible source of the effects, namely, an analog of the Hund rule in atomic physics. Consider a system of N pairwise interacting spins with the Hamiltonian $H = A\sum_{a\neq b}\mathbf{s}_a\cdot\mathbf{s}_b = A[\mathbf{S}^2 - Ns(s+1)]$. If the interaction strength A is a random variable with zero mean, then the ground state of the system has equal, $f_0 = f_{S_{\max}} = 1/2$, probabilities to have spin $S = 0$ or $S = S_{\max}$ (antiferromagnetism or ferromagnetism). A similar situation takes place in the degenerate pairing model where the pair creation, $P_0^\dagger$, pair annihilation, P_0, and particle number, N, operators form an SU(2) quasispin algebra. Then the eigenenergy is proportional to the pairing constant V_0 (the energy of a pair of fermions coupled to angular momentum $L = 0$) so that, for a random sign of V_0, the ground state quasispin will be 0 (unpaired state of maximum seniority) or maximum possible (fully paired state of zero seniority), on average in 50% of the cases. In the SU(3) model as well as in any model with a rotational spectrum $E_{\rm rot} = \mathbf{J}^2/2I$ the normal (inverted) bands will appear if the moment of inertia I takes positive (negative) values randomly.

For simplicity we limit ourselves here to a case of N identical fermions on a single-j shell which provides a generic framework for the extreme limit of strong residual interactions. Rotational invariance is preserved, so that all single-particle m states are degenerate. Within this space, the general two-fermion rotationally invariant interaction can be written as

$$H = \sum_{L\Lambda} V_L P_{L\Lambda}^\dagger P_{L\Lambda}\,, \tag{1}$$

where the pair operators with pair spin L and its projection Λ are defined as $P_{L\Lambda} = \frac{1}{\sqrt{2}}\sum_{mn} C_{mn}^{L\Lambda} a_n a_m$, and C are the Clebsch-Gordan coefficients. Because of Fermi statistics, only even L values are allowed in the single-j space. This fact was ignored in the attempt [8] to construct the random quasiparticle ensemble with identical distributions of

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the parameters V_L in the particle-particle channel and the parameters $\tilde{V}_K$ for the same interaction transformed to the particle-hole channel, $H \sim \sum_{K\kappa} \tilde{V}_K (a^\dagger a)_{K\kappa} (a^\dagger a)_{K\tilde{\kappa}}$ (the relation between the interactions in the two channels was discussed long ago by Belyaev [13], and served as a justification for an interpolating model "pairing plus multipole-multipole forces"). Since K can take both even and odd values, the number of parameters is different in the two representations, and $\tilde{V}_K$ cannot be independent if V_L are.

Assuming that the constants V_L are random, uncorrelated, and uniformly distributed between -1 and 1, we get the distribution f_J of the ground state spin J_0 shown in Figs. 1(a) and 1(c) for $N = 4$ and $N = 6$ at different values of j. We show by dotted lines the *a priori* distributions f_J^s based on the fraction of states of given J in the entire Hilbert space for given N. The overwhelming probability f_0 shows the same phenomenon in the uniform ensemble as observed earlier in Gaussian ensembles of V_L [8,9,11]. Further evidence of the dominance of $J_0 = 0$ configurations is given by the example, Fig. 1(b), for an odd number of particles, where excess of the ground state spin $J_0 = j$ is evidently related to the ground spin $J_0 = 0$ in the neighboring even system. The probability for the maximum spin $J_0 = J_{\max}$ is also strongly enhanced.

The effect for $J_0 = 0$ seems to exist already in a crude approximation modeling fermionic pairs by bosons. The commutation relations for the fermion pair operators are (L and L' are even)

$$[P_{L'\Lambda'}, P^\dagger_{L\Lambda}] = \delta_{L'L}\delta_{\Lambda'\Lambda} + 2\sum_{mm'n} C^{L'\Lambda'}_{m'n} C^{L\Lambda}_{nm} a^\dagger_m a_{m'} \,. \tag{2}$$

The second term in (2) is of the order N/Ω where Ω is the space capacity ($= 2j + 1$ in our case). It is small for a small number of fermions and for a nearly filled shell (because of the particle-hole symmetry). For intermediate occupation this term is not small but can be approximately substituted by its mean value (the monopole part with spin $K = 0$). Then, after a simple renormalization, $P_{L\Lambda}$ become bosonic operators. This is the assumption used in the original boson expansion techniques [14] and later in the interacting boson models: fermionic pairs $P_{L\Lambda}$ are substituted by bosons $b_{L\Lambda}$, and the Hamiltonian (1) becomes a sum of random bosonic energies $\sum_{L\Lambda} \omega_L n_{L\Lambda}$. The ground state in each realization corresponds to the condensation of the bosons into the single-boson states $|L\Lambda)$ with the lowest value of ω_L. For a given L, the many-boson states with different J allowed for the condensate are degenerate, but the value $L = 0$ is singled out by the obvious fact that for $\omega_0 = \min$ all degenerate states have total spin $J = 0$, while for the minimum boson energy ω_L at $L \neq 0$ any specific value of J, including $J = 0$, appears only in a small fraction of states. If all V_L have the same distribution, we expect $f_0^b \approx 1/k$ where k is a number of (equiprobable) values of L. All other values $J \neq 0$ appear with small probabilities $\sim 1/k^2$. This is demonstrated by Fig. 1(d) where the pattern is qualitatively similar to that in Figs. 1(a) and 1(c). The bosonic effect gives only a part (decreasing with increasing j) of the $J_0 = 0$ dominance observed for the fermions. Another argument against the dominance of the bosonic correlations is given in Fig. 1(c): after exact elimination of the monopole term ($V_{L=0} \equiv 0$), the picture does not significantly change although the value V_0 is now the lowest only in a small fraction, $\sim 2^{-(k-1)}$, of all cases (when all $V_{L\neq 0}$ are positive).

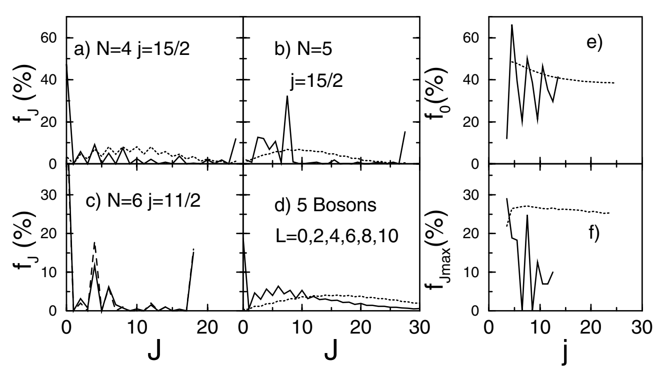

FIG. 1. The distribution of ground state angular momenta for various systems of N fermions of spin j (a)–(c). The bosonic approximation, f_J^b, is in panel (d). The dotted lines in (a),(b),(d) are the distribution of allowed J and the solid lines are the ensemble results. In (c) the dashed line is for $V_0 = 0$, i.e., no pairing. In (e) we have f_0 for $N = 4$ and different j; ensemble results (solid line), statistical theory (dotted line). (f) is similar to (e) with theoretical upper limit, dotted line, for $f_{J_{\max}}$.

In our opinion, the main effect comes from the statistical correlations of the fermions. They resolve the bosonic degeneracy in favor of the $J = 0$ and $J = J_{\max}$ ground states. In the strong mixing among nearly degenerate states, the eigenstates emerge as complicated chaotic superpositions. The only constraints left are the conservation laws for the particle number and total spin. The latter can be taken into account by the standard cranking approach. Thus, we model the system by the equilibrium Fermi gas with the occupation numbers n_m of individual orbitals characterized by the angular momentum projection m onto the cranking axis. The presence of the constraints creates a "body-fixed frame" and splits effective quasiparticle energies, although instead of the collective rotation around a perpendicular axis we have here a random coupling of individual spins with the symmetry (cranking) axis being the only direction singled out in the system [15]. Under the constraints

$$N = \sum_m n_m, \qquad M = \sum_m m n_m \,, \tag{3}$$

equilibrium statistical mechanics leads to the Fermi-Dirac distribution

$$n_m = [\exp(\gamma m - \mu) + 1]^{-1} \tag{4}$$

determined by the Lagrange multipliers of the chemical potential μ and cranking frequency γ; in the end the total projection M (equivalent to the K quantum number for axially deformed nuclei) is identified with the total spin J.

The quantities $\mu(N, M)$ and $\gamma(N, M)$ can be found directly from (3). At $M = 0$ we have $\gamma = 0$, so that the expansion in powers of γ allows one to study the most important region around $M = 0$; the power expansion is sufficient for all M except for the edges. With no cranking, one has the uniform distribution of occupancies $n_m^0 = \bar{n} = N/\Omega$. With the perturbational cranking, the occupation numbers are

$$n_m = \bar{n}\left[1 - \gamma m(1 - \bar{n}) + \frac{\gamma^2}{2}(m^2 - \langle m^2 \rangle)(1 - \bar{n}) \times (1 - 2\bar{n}) + \cdots\right]. \qquad (5)$$

Here $\langle m^2 \rangle = (1/\Omega)\sum_m m^2 = \mathbf{j}^2/3$, and terms of higher orders are not shown explicitly. The expectation value of energy in our statistical system can be written as

$$\langle H \rangle = \sum_{L\Lambda m_1 m_2} V_L |C^{L\Lambda}_{m_1 m_2}|^2 \langle n_{m_1} n_{m_2} \rangle. \qquad (6)$$

Neglecting the correlations between the occupation numbers, $\langle n_{m_1} n_{m_2} \rangle \approx n_{m_1} n_{m_2}$, we use the statistical result (5) and calculate the geometrical sums with the Clebsch-Gordan coefficients. Expressing the parameter γ in terms of the total spin $M \to J$, we come to the result including the terms of the second order in J^2,

$$\langle H \rangle_{N,J} = \sum_L (2L + 1) V_L [h_0(L) + h_2(L) J^2 + h_4(L) J^4], \qquad (7)$$

$$h_0(L) = \bar{n}^2, \qquad h_2(L) = 3(\mathbf{L}^2 - 2\mathbf{j}^2)/2\mathbf{j}^4 \Omega^2, \qquad (8)$$

$$h_4(L) = \frac{9}{40} \frac{(1 - 2\bar{n})^2 (3\mathbf{L}^4 + 3\mathbf{L}^2 - 12\mathbf{j}^2\mathbf{L}^2 - 6\mathbf{j}^2 + 8\mathbf{j}^4)}{(1 - \bar{n})^2 N^2 \Omega^2 \mathbf{j}^8}. \qquad (9)$$

The terms h_0 and h_2 can be also found directly from the $K = 0$ and $K = 1$ components of the interaction (1) in the particle-hole channel.

J_0 is determined by the ensemble distributions of $h_{2,4} = \sum_L (2L + 1) V_L h_{2,4}(L)$. For all realizations of the random interaction with $h_2 \geq 0$ and $h_4 > 0$, the ground state has spin $J_0 = 0$. If $h_2 > 0$ but $h_4 < 0$, one has a local energy minimum at $J = 0$ although there is a possibility to reach the absolute minimum at $J_{\max} = (1/2)N(\Omega - N)$. This will not happen if at $J = J_{\max}$ we still have $h_2 + J_{\max}^2 h_4 > 0$. Therefore the probability to have $J_0 = 0$, equals, in this approximation, to the integral of the probability $\mathcal{P}(h_2, h_4)$ over the region $h_2 > 0$, $h_4 > -(h_2/J_{\max}^2)$. Since h_4 is small, the result is close to that for the semiplane $h_2 > 0$, and f_0 is close to 50%. For a Gaussian distribution of the parameters V_L with zero mean and variances σ_L, the distribution of the linear combinations $h_{2,4}$ is again Gaussian, and the integral gives

$$f_0 = \frac{1}{4} + \frac{1}{2\pi} \arctan\left[\frac{A_{24} + A_{22}/J_{\max}^2}{\sqrt{A_{22}A_{44} - A_{24}^2}}\right], \qquad (10)$$

which is close to 1/2. Here we introduced the combinations of geometric factors weighted with the corresponding variances, $A_{pq} = \sum_L h_p(L) h_q(L) \sigma_L^2$.

The γ expansion fails for large momenta. However, the states with high M can be constructed exactly. For Figs. 1(e) and 1(f) we used our statistical approach near $J = 0$ in conjunction with the exact values in the end region $J = J_{\max}$ to improve the above result for f_0 and to get an upper bound for $f_{J_{\max}}$. Certainly, statistical theory predicts smooth average behavior only; irregularities of $f_J(j)$ should be considered separately. Figure 2 shows that the subset of the values of V_L leading, according to statistical theory, to $J_0 = J_{\max}$ agrees well with the empirical data.

Figure 3(a) shows a strong correlation of actual ground state energies E_0 for specific realizations of the ensemble

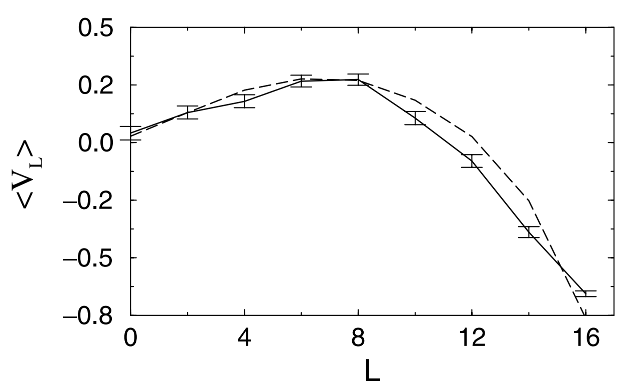

FIG. 2. Mean values of the interaction parameters $\langle V_L \rangle$, for ground states with spin $J_{\max}$ in the $N = 6$, $j = 17/2$ system; numerical simulations (solid line), statistical model predictions (dashed line).

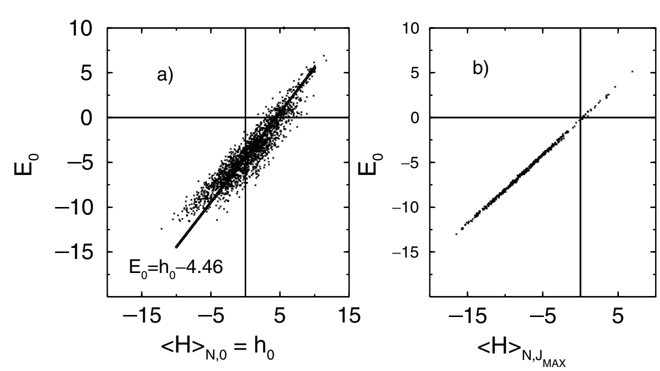

FIG. 3. Ground state energies E_0 for $J_0 = 0$ (a) and $J_0 = J_{\max}$ (b) vs predictions of the statistical model, Eq. (7).

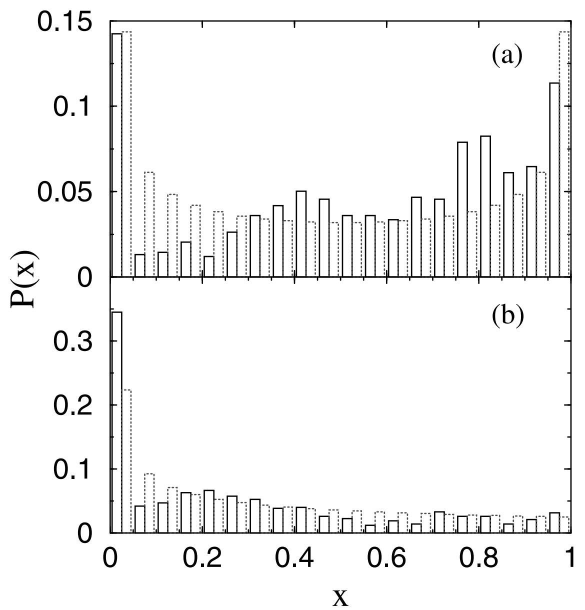

FIG. 4. The distribution of overlaps of $J_0 = 0$ ground states of the degenerate pairing model ($V_0 = -1$, $V_{L\neq 0} = 0$) with those for (a) random ensemble of $V_{L\neq 0}$ and $V_0 = -1$, (b) random ensemble of all V_L; $N = 6$ and $j = 11/2$ in both cases, and the dotted histogram is the predicted $P(x)$.

with statistical predictions (7). In the case of $J_0 = 0$, E_0 differs from $h_0 = \bar{n}^2 \sum_L (2L + 1) V_L$ essentially by a constant negative shift. This nonstatistical effect, as well as the distribution of gaps in the energy spectrum and other regularities found in [9] and confirmed by our calculations, is presumably due to the regular part of the dynamics related to $\langle V_L^2 \rangle$. The simplest description of this part can be reached with the aid of the boson expansion technique [14] and corresponds in fact to boson pairing. For $J_0 = J_{\max}$, Fig. 3(b), the ground state energy (7) is in one-to-one correspondence to the statistical predictions although the slope differs slightly from unity because of the use of the γ expansion. We hope to return to the discussion of nonstatistical phenomena elsewhere.

Although the energy spectra with random two-body interactions bear clear resemblance to the ordered spectra of pairing forces [9], the structure of the eigenstates is close to that expected for chaotic dynamics [10]. Figure 4(b) shows the distribution $P(x)$ of the overlaps $x = |\langle J = 0, \text{g.s.} | 0, p \rangle|^2$ of ground states with spin 0 obtained in the random ensemble with the ground state $|0, p\rangle$ for the degenerate pairing model, the latter corresponding to the case of fixed $V_0 = -1$, $V_{L\neq 0} = 0$. In the chaotic limit the wave functions are expected [4] to behave as random superpositions of basis states with uncorrelated components C uniformly spread over a unit sphere, $P(C) \propto \delta(\sum C^2 - 1)$. This is equivalent to the distribution of a single component $P(C_1) \propto (1 - C_1^2)^{(d-3)/2}$ where d is the space dimension. For $d \gg 1$, the distribution $P(C_1)$ is close to Gaussian whereas the overlaps $x = C_1^2$ obey the Porter-Thomas distribution. In the case of Fig. 4 ($N = 6$, $j = 11/2$) the dimension of the $J = 0$ space is small, $d = 3$, so that $P(C_1)$ is constant, and we expect $P(x) \propto 1/\sqrt{x}$. Another case, Fig. 4(a), corresponds to the overlap of $|0, p\rangle$ with the actual ground state for $V_0 = -1$, $V_{L\neq 0}$ random. Of course, here the completely paired state can appear as the ground state even for random strengths in the channels $L \neq 0$ which gives the peak at the overlap $x \to 1$. But the character of the distribution changes as well becoming effectively two dimensional: for $d = 2$, $P(x) \propto 1/\sqrt{x(1 - x)}$. This does not contradict the enhancement [9] of pair transfer between the adjacent ground states which is similar to the compound rotational band [7] in N space ("gauge rotation").

To conclude, we have shown that statistical correlations of fermions in a finite system with random interactions drive the ground state spin to its minimum or its maximum value. The effect is universal being related to the geometric chaoticity of the spin coupling of individual particles. This means that the dominance of 0^+ ground states in even-even nuclei may at least partly come from incoherent interactions rather than solely from coherent pairing. The structure of ground states with an "antiferromagnetic" ordering, $J_0 = 0$, is compatible with the predictions for chaotic dynamics. Quantitative relations between the geometric chaoticity and pure dynamic effects in finite many-body systems should be an interesting subject for further detailed studies.

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