Bifurcation in Viscoresistive MHD: The Hartmann Number and the Reversed Field Pinch

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A scaling approach to the simplest viscoresistive MHD model reveals that the Prandtl number acts only through the inertia term. When this term is negligible the dynamics is ruled by the Hartmann number H only. This occurs for the reversed field pinch dynamics as seen by numerical simulation of the model. When H is large the system is in a multiple helicity state. In the vicinity of $H = 2500$ the system displays temporal intermittency with laminar phases of quasi-single-helicity (SH) type. For lower H's two basins of SH are shown to coexist. SH regimes are of interest because of their nonchaotic magnetic field.

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In this Letter we consider the simplest viscoresistive nonlinear magnetohydrodynamics (MHD) model which is frequently used in the modeling of laboratory plasmas [1,2]. References [3,4] write the viscoresistive compressible MHD equation in the constant-pressure constant-density approximation as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \mathbf{J}), \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{J} \times \mathbf{B} + \nabla^2(\nu \mathbf{v}), \tag{2}$$

with $\mathbf{J} = \nabla \times \mathbf{B}$ and $\nabla \cdot \mathbf{B} = 0$.

Note that the fluid acceleration appearing in the left-hand side of Eq. (2) corresponds to the inertia force density, $\rho d\mathbf{v}/dt$, since the density has been dropped out from our model equations, due to the constant density approximation. Here time and velocity are normalized to the Alfvén time and velocity, respectively, and the other variables to macroscopic values: in these units η is the inverse Lundquist number, $\eta = \tau_A/\tau_R \equiv S^{-1}$, and ν corresponds to the inverse magnetic Reynolds number, $\nu = \tau_A/\tau_V \equiv R^{-1}$, for a scalar kinematic viscosity. These numbers represent a set of two dimensionless numbers relevant to our discussion. These two parameters may be combined in several ways to convert them into two different classical dimensionless parameters. A tradition coming from resistive MHD has given a leading role to the pair Lundquist number S and magnetic Prandtl number $P = \nu/\eta$. The first part of this Letter introduces a simple scaling approach to these equations which reveals that P is naturally combined with the Hartmann number $H = (\eta\nu)^{-1/2}$ in a modified set of coupled equations. These rescaled equations prove to be extremely useful in the frequent case in plasma physics where the impact of the inertia term fades away in Eq. (2). Then the rescaled equations show the dynamics is ruled by a unique dimensionless parameter: H. The importance of this number was first pointed out for magnetic confinement in Refs. [5–7].

The second part of this Letter proposes a new understanding of the turbulent-laminar transition in the reversed field pinch (RFP) based on an extended series of numerical simulations which are analyzed in reference to phase transition, bifurcation, temporal intermittency, and fluid dynamics theories. The RFP is a magnetic confinement device belonging to the family of the stabilized z-pinches like the tokamak, but working at higher toroidal current; this induces a process of magnetic self-organization leading to the reversal of the toroidal magnetic field in the outer part of the plasma [1,2,8]. The turbulent-laminar transition in the RFP corresponds to the passage from a multiple helicity (MH) to a single helicity (SH) state, so-called with reference to the number of helical components in the Fourier spectrum of the fields. This SH state has a sign of helicity opposite to that of the helical state in Taylor's theory of relaxation [9]. Numerical works in the early nineties indicated the possibility to induce such dynamical transitions in a RFP configuration by acting on the dissipation coefficients [3,4,10]. In contrast to the preceding results about this transition which proposed P as a control parameter at fixed S, we show, as an illustration to the first part of the Letter, that H is the true and unique control parameter in the physical regimes of interest. The turbulent-laminar transition may be viewed as a second order phase transition and we introduce a simple order parameter to characterize it.

We now want to focus on the case where the inertia term in Eq. (2) is small with respect to the two other ones. First consider the case where it may be neglected. As this limit includes that of high viscosity, we may suspect that normalizing time to the Alfvén time is not the most appropriate. When setting $t = \nu^a \eta^b \bar{t}$, parameters ν and η are present in the two limit equations only through their product for $a = -b = 1/2$. Therefore we apply to Eqs. (1) and (2) the rescaling:

$$t \longrightarrow \bar{t} = \sqrt{\frac{\eta}{\nu}}\, t, \tag{3}$$

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with the corresponding rescaling of velocity $\mathbf{v} \rightarrow \bar{\mathbf{v}} = \sqrt{(\nu/\eta)}\,\mathbf{v}$. This yields

$$\frac{\partial \mathbf{B}}{\partial \bar{t}} = \nabla \times (\bar{\mathbf{v}} \times \mathbf{B}) - \nabla \times (H^{-1}\mathbf{J}), \qquad (4)$$

$$P^{-1}\left[\frac{\partial \bar{\mathbf{v}}}{\partial \bar{t}} + (\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}}\right] = \mathbf{J} \times \mathbf{B} + \nabla^2(H^{-1}\bar{\mathbf{v}}), \qquad (5)$$

where H is the Hartmann number and P is the Prandtl number introduced above. The rescaled dynamics depends only on the value of the Hartmann number when the inertia term becomes negligible. This is shown here to be the case when studying the properties of the time averaged global quantities characterizing the RFP configurations. The benefit of the proposed transformation extends straightforwardly to more general and complete MHD modeling than our RFP-oriented model equations such as, for instance, working in vorticity representation, and including pressure and density evolution, nonuniform magnetic diffusivity, and kinematic viscosity; however isotropic transport coefficients are to be considered.

Previous studies pointed out H to be the unique intrinsic *physical* control parameter: (i) this was rigorously shown in Ref. [5] for the linear stability analysis of viscoresistive MHD modes in an incompressible plasma with a vanishing velocity field (see also Ref. [6] and references therein); (ii) it appeared through the combination $H^{-1/3}$ in Ref. [11] for a bifurcation approach of tearing modes in the magnetic slab configuration. Equations similar to Eqs. (4) and (5) are used for the convective instability of a horizontal fluid layer [12–14]. There the Prandtl number is also present through its inverse as a coefficient in the inertia term only. The Rayleigh number of the thermal convection system is the analog of the Hartmann number by combining kinematic viscosity and thermal conductivity through their product. For both systems the aspect ratio (R/a for a toroidal plasma, with R, a major and minor radius) is a *geometrical* control parameter [13,15,16]. Our system is expected to be sensitive also to the pinch parameter: $\Theta \equiv B_\theta(a)/\langle B_z \rangle$ (with θ, z the azimuthal and axial coordinates). It controls equilibrium bifurcations with a breaking of the axial symmetry, for example, in the case of the tokamak, in terms of the Kruskal-Shafranov ideal stability limit for external kink modes (see, for example, [1]); this limit is usually expressed as $q_a > 1$, [with $q_a \equiv aB_z(a)/RB_\theta(a)$], and may be written approximately in terms of the pinch parameter and of the aspect ratio as $\Theta < a/R$. Θ also rules bifurcations in simulations with Eqs. (1) and (2) for a tokamaklike configuration [6,7]. This dimensionless number is related to the strength of the driving on the system as it is proportional to the ratio of the total current to the magnetic flux. In this sense, the role of the temperature gradient in the Rayleigh number is analogous to that of the current in the pinch parameter.

Recent experimental observations of quasi-SH (QSH) states, mostly in RFX [17,18], the largest present RFP, have called for a more precise definition of the SH/MH transition. Laminar helical states are of interest for fusion studies because of the nonchaotic character of the pure SH magnetic field and therefore improved confinement properties may be expected for these regimes [17–19]. We now describe the results of our study based on the numerical simulation of the nonlinear model (1), (2), which correct and complement the previous bifurcation picture [3,10] in several respects. Two values of the Lundquist number are considered: $\eta^{-1} = 3.3 \times 10^3$, $\eta^{-1} = 3.0 \times 10^4$, corresponding to a choice of P in the intervals $[\frac{2}{3}, 10]$ and $[1, 5000]$. The plasma current and the toroidal magnetic flux are constant, and thus so is the pinch parameter, $\Theta = 1.9$. The RFP is simulated as a straight periodic cylinder with periodicity $2\pi R$, and the aspect ratio is $R/a = 4$. More details concerning the numerical modeling may be found in Ref. [4].

Figure 1 displays the kinetic energy, as the volume integral value, corresponding to the usual velocity field $\mathbf{v}$, and to the rescaled field $\bar{\mathbf{v}}$. This picture supports rescaling (3) and the Hartmann number as a good parameter to organize results. In fact, for the rescaled quantity, the points group as for a single value function of H, instead of lying on two separate curves. The numerical simulations reveal that the temporal intervals needed for good statistical averages scale as the viscoresistive time $\tau_{\rm vr} = P^{1/2}\tau_A$, consistently with the transformation (3).

The most unstable modes in RFP configurations correspond to $m = 1$. These modes are in general the largest ones in the system. They generate by nonlinear coupling the energy transfer to $m = 0$ and $m = 2, 3, \ldots$ modes [20,21]. On the contrary in the SH state only one $m = 1$ axial component, $n_{\rm SH}$, and its higher harmonics, $m/n = 1/n_{\rm SH}$, appear in the spectrum. When the system

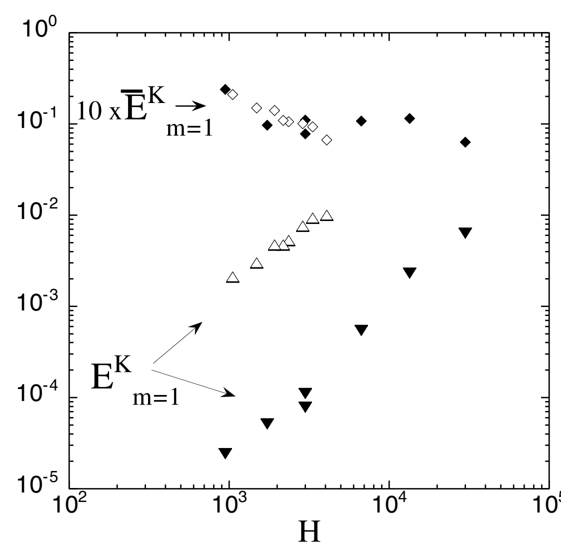

FIG. 1. Time averaged total kinetic energy of the spectral component $m = 1$ for the original velocity field, $\mathbf{v}$, and the rescaled one, $\bar{\mathbf{v}}$; triangles for $E^K_{m=1}$, diamonds for $10\,\bar{E}^K_{m=1} = 10\,PE^K_{m=1}$ (an amplification factor of 10 has been used to avoid superposition of points); open and black symbols, respectively, for $S = 3.3 \times 10^3$ and $S = 3.0 \times 10^4$.

settles in such a SH state starting from a MH condition, the $m = 0$ energy is seen to decay exponentially in time toward the pure helically symmetric magnetic SH configurations. Therefore the $m = 0$ mode energy $E^M_{m=0}$ represents an order parameter for the system which distinguishes the SH case from the MH one. Figure 2 plots $E^M_{m=0}$ vs H for the two previous values of the Lundquist number. This picture shows that on average our system reaches conditions where the impact of the inertia term in Eq. (5) becomes negligible and where H rules the qualitative dynamics as a unique control parameter independently of P even when it is of the order of 1. Indeed a direct estimate of the global power flow associated with the model equations yields, on average, values of the inertia term 2 orders of magnitude lower than the Lorentz and viscous ones. Furthermore Fig. 2 highlights a dynamical transition from a turbulent to a laminar regime in the interval $H = 2000-3000$. Above these critical values the system is in the MH state, and $m = 0$ modes reach their highest amplitude. At low values of the Hartmann number, below the transition interval, the system enters a laminar SH regime (as usual in nonlinear dynamics a high dissipation is favorable to laminar motion). This occurs when starting with a MH spectrum or with a SH spectrum with a small perturbation of MH type: these states are stable to three-dimensional perturbations. The SH regime corresponds to two basins of attraction characterized by helicity values: $m/n = 1/11$ and $m/n = 1/12$; both correspond to internal resonant modes, i.e., modes resonating inside of the region with a nonreversed magnetic field. As

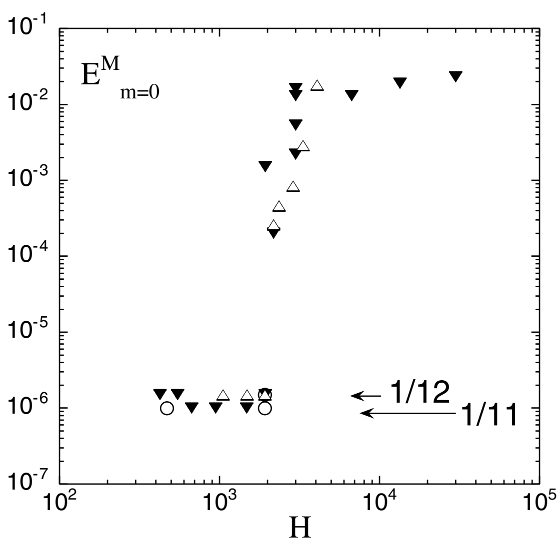

FIG. 2. Time averaged magnetic energies of the spectral component $m = 0$; open triangles for $S = 3.3 \times 10^3$, $P \in [\frac{2}{3}, 10]$; black triangles for $S = 3.0 \times 10^4$, $P \in [1, 5000]$; open circles for simulations where the initial condition is a slightly perturbed SH state; the other symbols refer to simulations started with fully developed turbulent states; for a convenient representation in the log-scale plot the vanishing SH energy is represented with a finite value, about $\sim 10^{-6}$, differing for the two helicities.

for the convective instability of a horizontal fluid layer [13], the existence of these two basins could be due to the finite value of R/a which prevents the system from developing the most unstable mode of the infinite R/a case, but selects one of the two adjacent ones. For a given Θ, an appropriate choice of the aspect ratio, might select the optimum one. In MH regime the system displays turbulent dynamics with sharper quasiperiodical relaxation events [1,2] the higher the H value. Macroscopic current sheet structures form in the plasma for values larger than $H_{SP} \approx 3.0 \times 10^4$: in this regime a Sweet-Parker dynamics is expected to dominate [4].

We now try to compare these values with the experimental estimates obtained for the dissipation coefficients. Present experiments are characterized by $S \sim 10^5-10^7$ (when using the Spitzer formula for the resistivity, which is a commonly accepted choice as motivated in Ref. [22] and references therein). Therefore the corresponding Prandtl number to get a value of the order of H_c is $P_c \equiv S^2/H_c^2 \sim 10^3-10^7$. Note that as P_c scales like η^{-2} and P scales like η^{-1}, an anomalous enhancement of resistivity would be favorable to reach SH states. A collisional estimate of $\nu_\perp$ would yield $P_\perp \sim 1$. However anomalous viscosity mechanisms are likely to be active in the RFP [22,23], such as those related to ion temperature gradient (ITG) driven instabilities, as discussed in a recent analysis of RFX data [22]. This was proposed in the past to explain the anomalous ion heating diagnosed in several RFPs [24].

When applied to experiments where QSH states were found in RFX [17,18], this ITG estimate yields $P \sim 100$ which is closer to the range where numerical SH states are obtained.

The dynamics in the transition region corresponds to a temporal intermittency [25] whose laminar phases are of a QSH type: the $m = 0$ modes may keep for long time intervals finite energy values in the range spanning a couple of orders of magnitude down by the typical MH conditions. In these cases the preferred helicities correspond to the two preferred values in SH states. As an example of these different dynamical situations, we show in Fig. 3 the temporal evolution of $(m = 1, n)$ modes for the various n's included in the computation. The two evolutions are obtained by starting the simulations with different initial conditions: in Fig. 3a, a perturbed $1/11$ single helicity condition is used, while in Fig. 3b a MH initial state is chosen. The H number is the same for the two cases: $H = 3000$. In the case of Fig. 3a the system switches to a $1/12$ QSH from a $1/11$ QSH after a short period of energy exchange between modes. Note how the preferred helicities of the system, corresponding to the two SH basins, may persist for long time intervals compared with the typical time scale of MH dynamics (Fig. 3b).

Long transients of this kind in the intermediate regime yield different values of $E^M_{m=0}$ over the finite duration of the simulations, as seen, for example, in Fig. 2 at $H = 3000$ where the black triangles give the value in correspondence of the different dynamical stages shown in Fig. 3. These

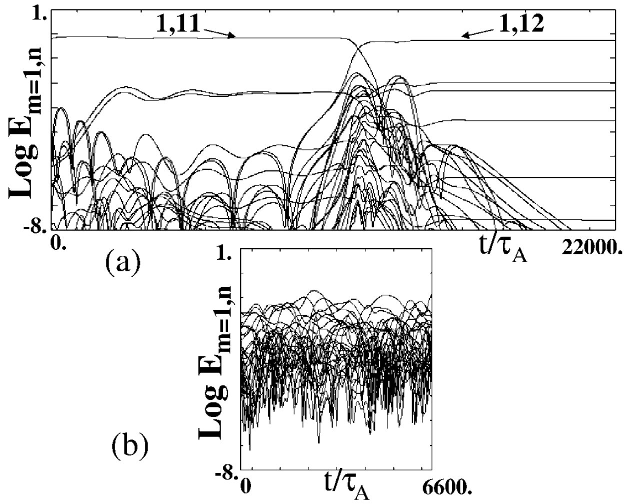

FIG. 3. Intermittent behavior in the transition region: temporal evolution of the magnetic energy for the $(m = 1, n)$ modes with the various n's included in the computation. The two evolutions are obtained by starting the simulations with different initial conditions at the same $H = 3000$ value. (a) shows the typical behavior of QSH regimes, (b) MH shows a typical MH turbulent regime.

transients were probably at the origin of the claim of the existence of a SH basin in parallel to the MH one in Ref. [10]. The existence of two basins of attraction and of an intermediate regime of temporal intermittency shows the existence of the bifurcation of a pair of stable fixed points in the dynamics [25]. We notice that the SH/MH transition is analogous to a second order phase transition where $E^M_{m=0}$ is the order parameter, H the control parameter, and where the intermediate QSH regime corresponds to the critical divergence of the correlation scales.

In conclusion the role of the Hartmann and Prandtl numbers in viscoresistive MHD and the description of the SH/MH transition in the RFP are clarified in several important respects: if P is not small, it has a vanishing role in controlling the dynamics, and H becomes the only control parameter of the nonlinear system; by using the $m = 0$ modes energy as an order parameter, a clear diagram is obtained for the SH/MH transition in the RFP. A threshold region showing intermittent periods of QSH is found which suggests the analogy with a second order phase transition. Pure SH states are the only stable state of the system when H is low enough. This supports the search for such pure and robust laminar regimes on the experimental side. Note that boundaries spoiled by magnetic field errors, such as in a real system, may force a QSH state from preventing the achievement of the orderly SH regime.

Future work is needed to assess the nature of the intermittent regime of the SH/MH transition, the impact on this transition of the aspect ratio, of the pinch parameter, and of transport physics. Efforts should also be devoted to the assessment of viscous effects, still an open issue in plasma physics (see, for example, the discussion in [5–7]). In this respect, anomalous contributions should also be considered for the RFP configurations [23]. Another issue for further investigation is related to the improvement of the boundary conditions: possibly forcing the right ones might enable one to reach the pure and robust laminar SH regime. Such a work will be important to assess the reactor relevance of the SH state of the RFP and of its regular magnetic field.

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