Quantum Limits on Optical Resolution

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We discuss the ultimate limit imposed by quantum fluctuations of light for resolution of fine details in optical images. For this purpose, we extend in the quantum domain the classical analysis of the object reconstruction, or superresolution, in terms of prolate spheroidal function basis. We derive the expression for ultimate resolution limit in the reconstructed object using an illumination of the full object plane by a multimode squeezed vacuum. We show that the gain in resolution using multimode squeezed light is maximum when the Shannon number of the imaging system is close to unity.

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Spatial behavior of nonclassical light, "quantum structures," and "quantum images" are presently the subject of active research [1,2]. This latest development in quantum optics casts a new light on such a well-known problem from classical optics as the ultimate limit of resolution in optical systems. A classical and well-known criterion of resolution was formulated at the end of the last century by Abbe and Rayleigh, and states that the optical resolution is limited by diffraction of the system pupil [3]. However, it is recognized now that modern photodetector arrays and CCD cameras allow us to determine the position and the displacement of a microscopic object with a precision much higher than the diffraction limit. Techniques for measuring displacement in the nanometer range have been successfully employed to detect deflection of glass fibers [4–6], microscopic phase objects [7], movement of biological, subcellular vesicles [8], measurement of ultraweak absorption using the mirage effect [9], or in atomic force microscopy [10]. All these measurements have a sensitivity which is ultimately limited not by diffraction but by the quantum fluctuations of the light beam used in the experiment. In a recent paper [11] it was shown that the use of multimode squeezed light could significantly improve the resolution beyond the standard quantum limit in a displacement measurement.

In this Letter we go one step further than the simple detection of position or displacement, and discuss quantum limits of resolution for restoring arbitrary details of an object in a diffraction-limited optical system. For simplicity we will consider a one-dimensional case.

The scheme of diffraction-limited coherent optical imaging is shown in Fig. 1. An object of finite size X is placed in the object plane y. The first lens L_1 performs the Fourier transform of this object into the Fourier plane ξ where a pupil of size d is located. The second lens L_2 performs the inverse Fourier transform and creates an image in the image plane x. The finite size of the pupil introduces a finite bandwidth in the transmission of the spatial frequencies by the system, so that the image is not an exact copy of the object.

The so-called "superresolution" techniques are able to restore object details beyond the Rayleigh limit [12–15] or equivalently to recover the object spectrum outside the band of the system. In the case of an object of finite size X (see Fig. 1), its Fourier transform in the pupil plane is an entire analytic function, and analytic continuation of the object spectrum outside the spatial-frequency band allows in principle for unlimited resolution. However, such a precise reconstruction of the object is obviously extremely sensitive to the noise in the detected image. The ultimate limit of superresolution is therefore determined by the quantum fluctuations of light in the image plane.

Let us introduce dimensionless variables $s = 2x/X, s' = 2y/X$, and the space-bandwidth product $c = \frac{\pi d X}{2\lambda f}$. In terms of these variables the transformation L of the classical object amplitude $a(s')$ into the image amplitude $e(s)$ reads

$$e(s) \equiv (La)(s) = \int_{-1}^{1} \frac{\sin[c(s - s')]}{\pi(s - s')} a(s')\, ds', \qquad -\infty < s < \infty. \tag{1}$$

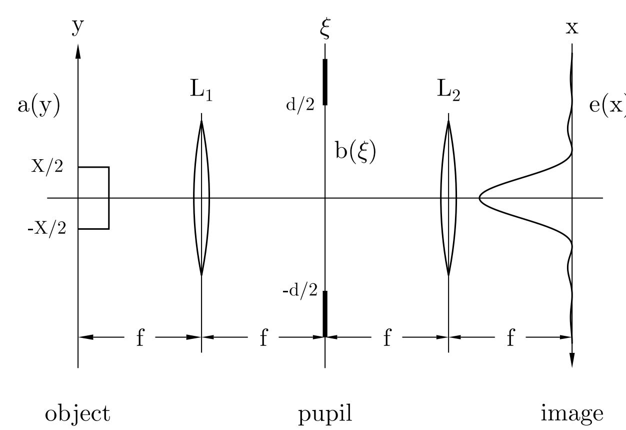

FIG. 1. Schematic of one-dimensional diffraction-limited coherent optical imaging.

The problem of reconstruction of the object $a(s')$ from a detected image $e(s)$ in the absence of noise is equivalent to inversion of the integral operator L. The operator $L^\star$ adjoint to L is given by [14]

$$(L^\star f)(s') = \int_{-\infty}^{\infty} \frac{\sin[c(s'-s)]}{\pi(s'-s)} f(s)\, ds, \qquad |s'| \le 1. \tag{2}$$

The product $A = L^\star L$ is the self-adjoint operator,

$$(Af)(s) = \int_{-1}^{1} \frac{\sin[c(s-s')]}{\pi(s-s')} f(s')\, ds', \qquad |s|, |s'| \le 1, \tag{3}$$

studied by Slepian and Pollak [16]. The orthonormal system of eigenfunctions of A is given by

$$\varphi_k(s') = \frac{1}{\sqrt{\lambda_k}} \psi_k(s'), \qquad |s'| \le 1, \tag{4}$$

where $\psi_k(s)$ are the prolate spheroidal functions [16,17], and λ_k are the corresponding eigenvalues. The functions $\varphi_k(s)$ form a basis in $L^2(-1,1)$ and may be considered as "elements of information" of the input object. The eigenvalues λ_k are an infinite set of real, positive numbers obeying $1 \ge \lambda_0 > \lambda_1 > \cdots > 0$. For small k the λ_k fall off slowly with k until the index reaches the critical value, $k = S$, called the Shannon number,

$$S = \frac{2c}{\pi} = \frac{dX}{\lambda f}, \tag{5}$$

beyond which the λ_k rapidly approach zero.

Using the fundamental properties of the prolate spheroidal wave functions,

$$\int_{-1}^{1} \frac{\sin[c(s-s')]}{\pi(s-s')} \psi_k(s')\, ds' = \lambda_k \psi_k(s), \qquad \int_{-\infty}^{\infty} \frac{\sin[c(s-s')]}{\pi(s-s')} \psi_k(s')\, ds' = \psi_k(s), \tag{6}$$

we obtain

$$L\varphi_k = \sqrt{\lambda_k}\, \psi_k, \qquad L^\star \psi_k = \sqrt{\lambda_k}\, \varphi_k. \tag{7}$$

Note that the functions $\psi_k(s)$ are defined on the real axis $-\infty < s < \infty$, and the functions $\varphi_k(s')$ on the interval $-1 \le s' \le 1$. Expanding the object amplitude over the functions $\varphi_k(s')$ and the image amplitude over $\psi_k(s)$, we can easily find the relation between the expansion coefficients of the object and the image. Indeed, since the functions $\varphi_k(s')$ form a complete orthonormal set in $[-1,1]$ we can write the object amplitude as

$$a(s') = \sum_{k=0}^{\infty} a_k \varphi_k(s'), \qquad |s'| \le 1, \tag{8}$$

with the coefficients a_k given by

$$a_k = \int_{-1}^{1} a(s') \varphi_k(s')\, ds'. \tag{9}$$

A similar expansion can be written for the image amplitude in terms of functions $\psi_k(s)$,

$$e(s) = \sum_{k=0}^{\infty} e_k \psi_k(s), \qquad -\infty < s < \infty. \tag{10}$$

Substituting these expansions into Eq. (1) and using the first of Eqs. (7) we obtain the following relation between a_k and e_k:

$$e_k = \sqrt{\lambda_k}\, a_k. \tag{11}$$

Let us denote by $\tilde{a}_k$ the expansion coefficients of the object reconstructed from the measured image $e(s)$. In the absence of noise these coefficients are obtained by dividing the image coefficients e_k by $\sqrt{\lambda_k}$. Thus, for a noise-free image the object reconstruction can be performed exactly, $\tilde{a}_k = a_k$, i.e., without resolution limit. The ultimate accuracy in the determination of $\tilde{a}_k$ will therefore be determined by quantum fluctuations in the measurement of e_k.

In the quantum theory the object amplitude $a(s')$ and the image amplitude $e(s)$ become operators obeying the standard commutation relations (see, for example, Ref. [1]). We can use Eqs. (8) and (10) now, treating the expansion coefficients a_k and e_k as photon annihilation operators. The operators a_k in the object plane obey the following commutation relations:

$$[a_k, a_l^\dagger] = \delta_{kl}, \qquad [a_k, a_l] = 0. \tag{12}$$

The same commutation relations must be satisfied by the image coefficients e_k. However, Eq. (11) does not preserve the commutation relations (12). The reason for this is that the classical imaging equation (1) takes into account only nonzero field amplitude in the region $|s'| \le 1$ of the object plane. The rest of this plane $|s'| > 1$ is ignored because there the classical field amplitude is zero. In the quantum theory this region must be taken into account to guarantee the conservation of the commutation relations.

To include the region $|s'| > 1$ in the object plane into the theory we introduce another set of functions:

$$\chi_k(s') = \frac{1}{\sqrt{1-\lambda_k}} \psi_k(s'), \qquad |s'| > 1. \tag{13}$$

Using the following properties of the prolate spheroidal wave functions,

$$\int_{-\infty}^{\infty} \psi_k(s)\psi_l(s)\, ds = \delta_{kl}, \qquad \int_{-1}^{1} \psi_k(s)\psi_l(s)\, ds = \lambda_k \delta_{kl}, \tag{14}$$

it is easy to show that functions $\chi_k(s')$ form a complete orthonormal set in the region $|s'| > 1$. Therefore, we can use them as a basis for expansion of the field outside the object.

Now the operator of the total field in the object plane, $-\infty < s' < \infty$, can be written as

$$a(s') = \sum_{k=0}^{\infty} a_k \varphi_k(s') + \sum_{k=0}^{\infty} b_k \chi_k(s'), \quad (15)$$

where the operators b_k satisfy the commutation relations similar to Eq. (12). Substituting the expansion (15) into Eq. (1) we obtain the following relation between the coefficients in the object and the image plane,

$$e_k = \sqrt{\lambda_k}\, a_k + \sqrt{1 - \lambda_k}\, b_k. \quad (16)$$

It is easy to verify that this transformation preserves the commutation relations of the operators, $[a_k, a_l^\dagger] = [b_k, b_l^\dagger] = [e_k, e_l^\dagger] = \delta_{kl}$.

Equation (16) is completely equivalent to the transformation performed by a beam splitter. Indeed, if we consider the operators a_k and b_k as the photon annihilation operators in the modes defined by prolate spheroidal waves incoming to the beam splitter with the amplitude transmission coefficient $\sqrt{\lambda_k}$ and the reflection coefficient $\sqrt{1 - \lambda_k}$, then e_k is the photon annihilation operator in the kth mode of the transmitted wave.

From Fig. 1 one may think that the vacuum fluctuations coming from the region $|\xi| > d/2$ in the Fourier plane outside the pupil should also be taken into account. Indeed, when treating the field in the Fourier plane as an operator we must include the contribution from this region into the resulting field in the image plane. However, the advantage of expansion (10) is that the field from this region does not contribute to the expansion coefficients e_k of the image because it is orthogonal to the prolate spheroidal wave functions. This property was pointed out by Bertero and Pike in [14] for the out-of-band classical noise and remains valid in the quantum theory, as we will prove elsewhere.

We will assume that we use a homodyne detection technique of the image that allows us to register any of the quadrature components of the field $e(s)$,

$$e(s) = e_1(s) + ie_2(s). \quad (17)$$

Using Eq. (10) we can express the variances of the expansion coefficients e_{1k} and e_{2k} of these quadrature components through the variances of coefficients a_{1k}, a_{2k} and b_{1k}, b_{2k} in the object plane,

$$\langle(\Delta e_{\mu k})^2\rangle = \lambda_k \langle(\Delta a_{\mu k})^2\rangle + (1 - \lambda_k)\langle(\Delta b_{\mu k})^2\rangle, \quad (18)$$

with $\mu = 1, 2$ for corresponding quadratures. The fluctuations of the expansion coefficients of the reconstructed object are obtained as follows:

$$\langle(\Delta \tilde{a}_{\mu k})^2\rangle = \frac{\langle(\Delta e_{\mu k})^2\rangle}{\lambda_k}. \quad (19)$$

Let us assume that the field in the object plane is in a multimode coherent state at any point $|s'| \leq 1$. Since there is no light outside the region $[-1, 1]$, this corresponds to the vacuum state of all operators b_k. In this case the variances of the coefficients $a_{\mu k}$ and $b_{\mu k}$ in the object plane are equal to $\langle(\Delta a_{\mu k})^2\rangle = \langle(\Delta b_{\mu k})^2\rangle = \frac{1}{4}$, and the variances of the coefficients for the reconstructed object read

$$\langle(\Delta \tilde{a}_{\mu k})^2\rangle = \frac{1}{4\lambda_k}. \quad (20)$$

As the eigenvalues λ_k become rapidly very small for $k > S$, the corresponding variance will become very large which will forbid a precise determination of the coefficient $\tilde{a}_{\mu k}$. Therefore, formula (20) sets the standard quantum limit in reconstruction techniques used in superresolution.

Since imaging an equation in the form (16) is equivalent to the transformation of two fields by a beam splitter it gives us an idea of how to reduce the fluctuations in the reconstructed object below the standard quantum limit. It is well known that such an improvement can be achieved by illuminating the open port of the beam splitter by a squeezed vacuum with properly chosen squeezed quadrature. Since in our case the role of such an open port is played by the region $|s'| > 1$ outside the object, we expect that superresolution beyond the standard quantum limit can be achieved by illuminating this region by a multimode squeezed vacuum with a light spot much larger than the size of the object. Such a light can be produced by a traveling-wave optical parametric amplifier or a degenerate optical parametric oscillator below threshold in a confocal cavity [1].

We may expect that an even better result for the object reconstruction can be obtained when not only the area outside the object but the object itself is illuminated by multimode squeezed light with nonzero mean amplitude. However, to use the advantage of such illumination we have to make sure that squeezing in the incoming light is not destroyed by absorption in the object. Therefore, squeezed light illumination should be advantageous for pure phase or for weakly absorbing objects.

In practice, one can use a single source of multimode squeezed vacuum with a large transverse area and mix it with a coherent light wave in the central part, illuminating the object using a weakly reflecting small size mirror. To evaluate the fluctuations of the reconstructed object in this case we assume for the sake of simplicity that squeezing has infinitely large spatial bandwidth, i.e., is the same for all coefficients a_k and b_k, $\langle(\Delta a_{\mu k})^2\rangle = \langle(\Delta b_{\mu k})^2\rangle = \frac{1}{4}e^{\pm 2r}$, where the "$-$" sign corresponds to the squeezed quadrature $\mu = 1$, and the "$+$" sign to the stretched one, $\mu = 2$, and r is the squeezing parameter. This gives

$$\langle(\Delta \tilde{a}_{\mu k})^2\rangle = \frac{1}{4\lambda_k} e^{\pm 2r}. \quad (21)$$

It follows from this equation that the variance of the reconstruction coefficients in the squeezed quadrature component $\tilde{a}_1$ can be significantly reduced below the standard quantum limit for large squeezing, $r \gg 1$. For $r = 0$ we recover the standard quantum limit of Eq. (20).

To estimate the resolution length D (the smallest object detail reconstructed from the image) from Eq. (20) and its improvement obtained using Eq. (21) we will use the arguments similar to Ref. [14]. For definiteness we shall consider an amplitude object. Let us define the signal S and the noise B related to the reconstructed object as

$$S = \int_{-1}^{1} \langle \tilde{a}_1(s') \tilde{a}_1(s') \rangle \, ds', \qquad B = \int_{-1}^{1} \langle (\Delta \tilde{a}_1(s'))^2 \rangle \, ds'. \tag{22}$$

These quantities can be evaluated as $S = \langle N \rangle$, where $\langle N \rangle$ is the total number of photons in the object, and

$$B \simeq \frac{e^{-2r}}{4} \sum_{k=0}^{Q} \frac{1}{\lambda_k} \simeq \frac{e^{-2r}}{4} \frac{1}{\lambda_Q}. \tag{23}$$

Here Q is the index of the highest eigenfunction in the reconstructed object. In the second equality we have used the fact that λ_Q is the smallest eigenvalue and estimated the sum as $1/\lambda_Q$. Setting the signal-to-noise ratio equal to unity we can find the smallest eigenvalue λ_Q that can be recovered from the image measurement,

$$\lambda_Q \simeq \frac{e^{-2r}}{4 \langle N \rangle}. \tag{24}$$

Using the tables of eigenvalues λ_k [17] or calculating them numerically we can evaluate the index Q of the highest eigenfunction $\varphi_Q(s)$ with a precisely known coefficient in the expansion of the reconstructed object. Knowing that $\varphi_Q(s)$ has exactly Q zeros on the interval $[-1, 1]$ we can estimate the resolution length D as

$$D \simeq X/(Q + 1) = R\left(\frac{S}{Q + 1}\right), \tag{25}$$

where $R = X/S = \lambda f/d$ is the Rayleigh resolution length. From this equation we can interpret the number $Q + 1 \equiv M$ as an effective number of degrees of freedom in the object which, as follows from Eq. (24), is a function of the total photon number $\langle N \rangle$ and the squeezing parameter r. The case of superresolution corresponds to $D < R$ or, equivalently, $M > S$. For classical noise the dependence of M on the classical signal-to-noise ratio was studied in [14,15] with an important conclusion that significant superresolution can be achieved for small values of the Shannon number S. Since our Eq. (24) is an extension to the quantum domain of analogous equation studied in Refs. [14,15] we conclude that a similar result holds true for quantum fluctuations.

A scanning optical microscope is operated in a small Shannon number configuration [18]. It is therefore a good candidate for a practical application of our analysis. We will consider it in more detail in a forthcoming paper together with the influence of the finite pixel size in the image plane and the finite coherence area of squeezed light in the object plane.

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