

Two-Body Random Ensembles: From Nuclear Spectra to Random Polynomials

Dimitri Kusnezov

Center for Theoretical Physics, Sloane Physics Laboratory, Yale University, New Haven, Connecticut 06520-8120
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The two-body random ensemble for a many-body bosonic theory is mapped to a problem of random polynomials on the unit interval. In this way one can understand the predominance of 0^+ ground states, and analytic expressions can be derived for distributions of lowest eigenvalues, energy gaps, density of states, and so forth. Recently studied nuclear spectroscopic properties are addressed.

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The origins of spectroscopic properties of nuclei have received renewed attention recently in the context of the two-body random ensembles [1–3]. These studies provide an understanding of which nuclear properties are robust, depending only on the model space (one- and two-body interactions), and which depend on specific strengths of interactions within the space. The 0^+ ground state is an example of a robust feature. The starting point for analyses of two-body random ensembles (TBREs) are Hamiltonians of the form $H = \sum_k \epsilon_k c_k^+ c_k + \sum_{ijkl} v_{ijkl} c_i^+ c_j^+ c_k c_m$, where c_k^+, c_k represent boson [4] or fermion [5] creation/annihilation operators for a state k , and the coefficients ϵ_k, v_{ijkl} are taken as Gaussian random variables once certain physical constraints are imposed, such that H commutes with the generators of total spin, isospin, and so forth. The important distinction between the TBRE and the conventional

Gaussian orthogonal ensemble (GOE) description of many-body Hamiltonians is that the latter does not include correlations between Hilbert subspaces of different quantum numbers, which are essential to understanding ground state and low energy spectroscopic properties of nuclei. Although studied for some time, very little is known analytically about the bosonic TBRE [1,4].

We consider a bosonic model which can be treated analytically, while retaining salient features of more complex theories. The U(4) vibron model [6] consists of two type of bosons, $J^\pi = 0^+, 1^-$, and is used to describe the rotations and vibrations of diatomic molecules. In contrast to the fermionic U(4) problem of particles in the $j = 3/2$ shell, the Hilbert space of the bosonic theory can be arbitrarily large. We will see that this model describes many recently observed nuclear properties in the U(6) TBRE [1,3]. The Hamiltonian is [6,7]

$$H = \frac{1}{N} (\epsilon_s s^+ s + \epsilon_p p^+ \cdot \tilde{p}) + \left\{ \frac{c_0}{2} [p^+ p^+]^{(0)} \cdot [\tilde{p} \tilde{p}]^{(0)} + \frac{c_2}{2} [p^+ p^+]^{(2)} \cdot [\tilde{p} \tilde{p}]^{(2)} + \frac{u_0}{2} [s^+ s^+]^{(0)} \cdot [s s]^{(0)} \right. \\ \left. + \frac{v_0}{2\sqrt{2}} ([s^+ s^+]^{(0)} \cdot [\tilde{p} \tilde{p}]^{(0)} + \text{H.c.}) + \frac{u_1}{2} [s^+ p^+]^{(1)} \cdot [\tilde{s} \tilde{p}]^{(1)} \right\} / N(N-1), \quad (1)$$

where $s^+(s)$ and $p_\mu^+(\tilde{p}_\mu = -p_{-\mu})$ are the spherical-tensor creation (annihilation) operators for states with $J^\pi = 0^+$ and 1^- (projection $\mu = 0, \pm 1$), respectively. The square brackets indicate angular momentum couplings and dots indicate scalar products. Since the matrix elements of the one- and two-body interactions are proportional to N and $N(N-1)$, scaling allows all coefficients to be Gaussian random numbers of unit variance. The random Gaussian variables are grouped into a vector $x = (c_0, c_2, u_0, u_1, v_0, \epsilon_s, \epsilon_p)$. The matrix elements of H are well known in the vibrational basis $|Nn_p Jm\rangle$, where N is the total number of bosons, $n_p = 0, 1, \dots, N$ is the number of $J^\pi = 1^-$ bosons, and $J = n_p, n_p - 2, \dots, 1$ or 0 is the total angular momentum of the many-body state ($m = -J, \dots, J$ is omitted since it adds a trivial degeneracy.) We also note that even (odd) J states have only even (odd) values of n_p , resulting in an odd/even effect, depending on the choice of N . While this effect is easily treated, we will focus only on even N to simplify the presentation.

When the dimension of the Hilbert space is large, which is typical of many-body problems, a common approach to diagonalization is Lanczos's method. Here successive iterations of the Hamiltonian on an arbitrary trial wave function Ψ_0 are performed to reduce the Hamiltonian to a tridiagonal form. For a given spin J of an N -body state, we label the basis by $n_p = J, J+2, \dots, N-1$ or N . Clearly the dimension of the $J=0$ Hilbert space is maximum, while that of $J=N$ is minimum. While it has been argued for some TBRE's that the dimensionality or the width of the lowest eigenvalue distribution is responsible for the preponderance of 0^+ ground states, we will see that this is not the origin here.

We choose the trial Lanczos state to be that with $n_p = J$. By enumerating the states by an index $k = (n_p - J)/2$, the Hamiltonian (1) assumes the tridiagonal form $H\Psi_k = \beta_k \Psi_k + \alpha_{k-1} \Psi_{k-1} + \alpha_{k+1} \Psi_{k+1}$, where α_k, β_k are the matrix elements of H in this basis. The study of tridiagonal matrices is intimately linked to orthogonal polynomials

and their recursion relations, where k might represent the order of the polynomial [8]. In particular, the family of recursively generated, real polynomials are related to the characteristic polynomial $D_i(E) = \det(\mathcal{H}_i - E \cdot 1_i)$ of the $i \times i$ tridiagonal matrix \mathcal{H}_i . Consequently the zeros of the polynomials D_i are related to the eigenvalues of \mathcal{H}_i . There are several theorems which have developed bounds for the zeros of D_i as well as expressions for the extreme eigenvalues which we can use to derive properties of the TBRE [8,9].

We consider two cases here: large N and $N = 2$. For $N = 2$, the order of the interaction (two-body) is equal to the number of particles, and we expect to recover the GOE. The only allowed states are $J = 0, 1, 2$. The $J = 1, 2$ Hilbert spaces are one-dimensional, while $J = 0$ is 2D. It can be readily checked that the density of $J = 0$ states gives the Wigner semicircle (i.e., the 2×2 matrix can be expressed as one with GOE measure), while the $J = 1, 2$ are always Gaussian. The same is true for $J = J_{\max} = N$ for any N , since the Hilbert space for the maximum spin states is always one dimensional so that the density of states is purely Gaussian [Fig. 1(b)].

In the following, we will use the large N limit (typically in molecules $N \sim 100$ [6,7], but $N = 8$ calculations already agree with our analytic predictions below). We next define $z = n_p/N$ and $j = J/N$ and construct the functions $\alpha(z, j)$, $\beta(z, j)$ from the analytic matrix elements of the tridiagonal matrix, $\beta(z, j) = \gamma_1 z^2 + \gamma_2 z + \gamma_{3,j}$, $\alpha(z, j) = \gamma_4 z(1 - z) + o(1/N)$, where γ_k are linear combinations of the random coefficients x . α and β

are the off-diagonal and diagonal matrix elements, respectively. Then, in the large N limit, the lowest eigenvalue for each spin has the form [8,10]

$$E_{\min}^j = \inf_z [f_j(z)], \quad f_j(z) = \beta(z, j) - 2|\alpha(z, j)|. \quad (2)$$

In terms of the parameters of H in Eq. (1) and the matrix elements of the interactions [6,7], we can express $f_j(z) = az^2 + bz + d_j$, where $z \in [j, 1]$ and

$$a = r_1 \cdot x = \left(\frac{c_0}{6} + \frac{c_2}{3} + \frac{u_0}{2} - u_1 + \frac{|v_0|}{\sqrt{6}} \right) \frac{N}{N-1},$$

$$b = r_2 \cdot x = \varepsilon_p - \varepsilon_s + (u_1 - u_0 - |v_0|/\sqrt{6}) \frac{N}{N-1},$$

$$d_j = r_3 \cdot x = \varepsilon_s + u_0/2 + j^2(c_2 - c_0)/6. \quad (3)$$

To check this approach, consider only the off-diagonal interactions ($\beta = 0$). Then we expect for small J that the minimum energy approaches $E_{\min}^j = -|v_0|N/4\sqrt{6}(N-1)$. The numerical value, denoted E_{calc}^j , can be seen to converge to this result. For $J = 0, 1, 2$, $E_{\text{calc}}^j/E_{\min}^j = 1.06, 1.05, 1.03$ for $N = 20$ and $1.01, 1.01, 1.01$ for $N = 80$. Hence the desired spectral properties of H can be recast in terms of random polynomials on the unit interval. This should be generally true for any bosonic or fermionic theory since the Lanczos approach can be applied to either. The main effort is to determine whether there is any approximate analytic behavior of the functions α, β , although it is suggested to be generally true [10].

Many general properties of random or Kac polynomials are known, but typically for higher order functions which are otherwise unrestricted [11]. To understand the properties of this TBRE, we first compute the distribution of coefficients. Since the variables x are taken as Gaussian with measure $P(x) \propto \exp[-\sum_k x_k^2/2]$, one can compute the distribution of coefficients of the random polynomial with $\mathcal{P}(a, b, d_j) \propto \int dx P(x) \delta(a - r_1 \cdot x) \delta(b - r_2 \cdot x) \delta(d_j - r_3 \cdot x)$. Integrating yields

$$\mathcal{P}(a, b, d_j) \propto \exp[-\frac{1}{2} J^T M^{-1} J], \quad J = (a, b, d_j). \quad (4)$$

Here $M_{\alpha\beta}^{-1} = (2 \det M)^{-1} \epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\epsilon} (\delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl}) \times r_{a,i} r_{b,j} r_{c,k} r_{d,l}$ is the inverse of $M_{\alpha\beta} = r_\alpha \cdot r_\beta$, and $r_{a,i}$ is the i th component of r_a .

As in the U(6) model [1], the frequency of a ground state 0^+ is approximately 70% (for even N : 70.5%, 71.7%, 72.3%, 72.5% for $N = 8, 16, 32, 64$, respectively). To understand this, we first compute where the minima $z = z_0$ of $f_0(z)$ are located. Integrating the location of the minima over the distribution of coefficients (4), we find the distribution of minima to be:

$$P(z_0) = \frac{1}{2\pi} \frac{\sqrt{r_1^2 r_2^2 - (r_1 \cdot r_2)^2}}{r_1^2 z_0^2 + (r_1 \cdot r_2) z_0 + r_2^2/4}. \quad (5)$$

In Fig. 1(a) we compare this function to results from 10^6 numerical diagonalization of Eq. (1) for $N = 8, 16, 32$,

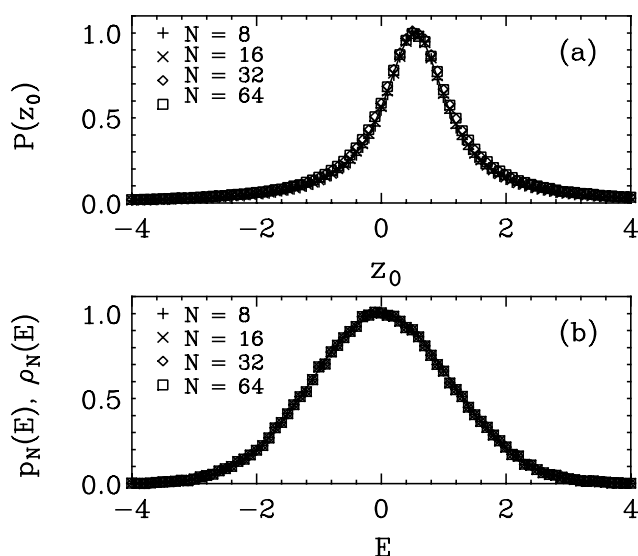


FIG. 1. (a) Theoretical distribution (5) of extrema of $f_0(z)$ (solid line) compared to numerical TBRE results (symbols). (b) Theoretical eigenvalue distribution for highest spin states $p_N(E)$ [solid line; Eq. (8)] are seen to be Gaussian and agree with numerical results (symbols). This is also the level density $\rho_N(E)$ of $J = N$ states for any N . (In this and the following figures, the vertical scale is arbitrary.)

and 64 bosons. We estimate the frequency of 0^+ ground states by first understanding where in $z \in [0, 1]$ the minima are located. For 50% of the cases, $a < 0$, so $f_0(z)$ is an inverted parabola, and its minimum is at $z = 0$ or 1, each occurring with equal probability (25%). For the remaining 50% of the cases, $a > 0$. For these cases, the probability of having the minimum in $z \in (0, 1)$ is

$$\frac{1}{2} \int_0^1 P(z_0) dz_0 = \frac{1}{2\pi} \tan^{-1} \left[\frac{2\sqrt{r_1^2 r_2^2 - (r_1 \cdot r_2)^2}}{r_2^2 + 2r_1 \cdot r_2} \right], \quad (6)$$

which has a value of 22%, leaving 14% at $z = 0$ and 14% at $z = 1$. Next we ask when the 0^+ state is a global minimum (over all J). At $z = 0$, only 0^+ states are allowed (since a state $J > 0$ has $n_p \geq J$, hence $z > 0$). Hence, for at least 39% of the cases, 0^+ is the ground state. At $z = 1$, all even J are allowed, and, from (2) and (3), $E_{\min}^j - E_{\min}^0 = j^2(c_2 - c_0)/6$. Since c_0, c_2 are Gaussian random numbers, half of the time 0^+ is the ground state ($E_{\min}^0 < E_{\min}^j$), and during the other half, is the state of maximum j , or $J = N$. So, at $z = 1$, 19.5% of the ground states are 0^+ and 19.5% are $J = N$. For $z \in (0, 1)$, we note that $J = 0$ has even n_p and hence is only allowed at half of the points. Consequently, of the 22% of the minima here, no more than 11% can be 0^+ ground states. Therefore we estimate a roughly 69.5% frequency of 0^+ ground states, in very good agreement with observations.

Since it is more likely to find the minima of $f_j(z)$ on the boundary than inside, we can compute distribution functions of interest by restricting attention to the edges $z = j$ and $z = 1$. While this is approximate, it does yield predictions which agree very well with the molecular U(4) and nuclear U(6) TBREs. Averaging $\int_z [f_j(z)]$ over $P(a, b, d_j)$ yields the distribution of lowest eigenvalues, denoted $p_J(E)$. These have the form

$$p_0(E) \propto \exp(-E^2/2r_3^2) \operatorname{erfc}(EA_1) + A_3 \exp(-E^2/2R^2) \operatorname{erfc}(EA_2), \quad (7)$$

$$p_{J_{\max}}(E) \propto \exp(-E^2/2R^2), \quad (8)$$

where $R^2 = (r_1 + r_2 + r_3)^2 = 1.07$, $A_1 = (r_1 \cdot r_3 + r_2 \cdot r_3)/(hr_3) = 0.663$, $A_2 = (R^2 - r_1 \cdot r_3 - r_2 \cdot r_3)/hR = 0.632$, $A_3 = hr_3/(R\sqrt{2} \det M) = 0.423$, and $h^2 = 2[r_3^2(r_1^2 + 2r_1 \cdot r_2 + r_2^2) - (r_1 \cdot r_3 + r_2 \cdot r_3)^2]$ (evaluated at $N = 64$). For other values of J , one can readily derive the general form which is a linear combination of two terms similar to (7). In Fig. 2(a) we compare (7) (solid lines) to a Gaussian (dashed lines) and to results from diagonalization of H for selected N . The low energy excess is readily described, and the results are not Gaussian for any N , in contrast to results from the dilute limit [4]. In Fig. 2(b) we compare the same function with slightly modified parameters ($r_3 = \sqrt{2}$ and $A_1 = 3/4$) and see that this functional form can readily account for the observed asymmetry in the U(6) results [1]. The results for maximum J are shown in Fig. 1(b). Since this

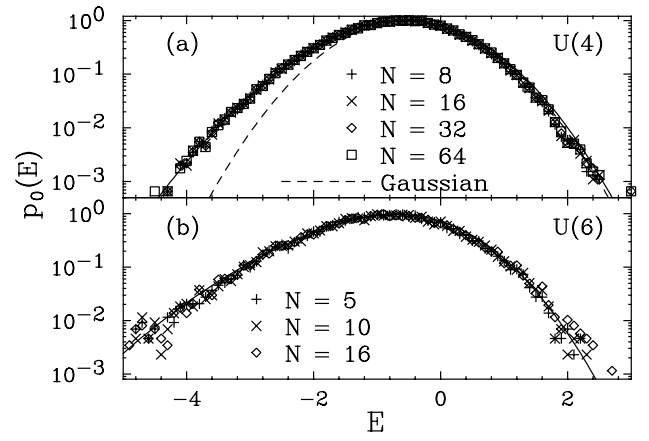


FIG. 2. (a) Lowest eigenvalue distribution (7) for $J^\pi = 0^+$ states (solid line) compared to numerical TBRE calculations. The Gaussian is for reference. (b) Same as (a) but for the nuclear case.

Hilbert space is 1D, the distribution also corresponds to the level density.

It is interesting to use $f_j(z)$ to estimate the level density $\rho_J(E)$ for states of spin J by averaging this over the space of random polynomials. For $J = J_{\max}$, we trivially recover (8). For $J = 0$ we find

$$\rho_0(E) = \int_0^1 dz \frac{1}{\sqrt{2\pi g(z)}} \exp\left[-\frac{E^2}{2g(z)}\right], \quad (9)$$

where $g(z) = Z^T M Z$, $Z = (z^2, z, 1)$, which agrees well with calculations (Fig. 3). $\rho_0(E)$ has moments:

$$\langle E^{2n} \rangle_{J=0} = \frac{2^n}{\pi} \Gamma(n + 1/2) \int_0^1 g(z)^n dz. \quad (10)$$

One can see that the shape of the level density is a superposition of Gaussians of varying width. The width of the 0^+ density of states is $\langle E^2 \rangle_{J=0} = 0.784$, while for the maximum spin $\langle E^2 \rangle_{J=N} = 1.137$. In this model we see that there is no direct relation between the widths of the distributions and the probability of having a 0^+ ground state, as has been conjectured in other TBREs [1,2].

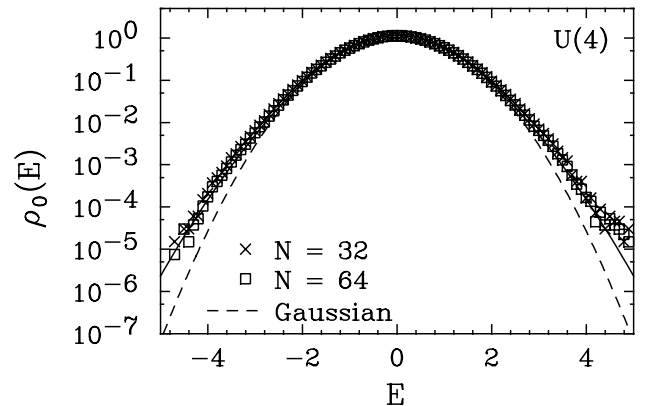


FIG. 3. Theoretical density of states (9) for 0^+ states (solid line) compared to a Gaussian, and numerical TBRE calculations for selected N (symbols).

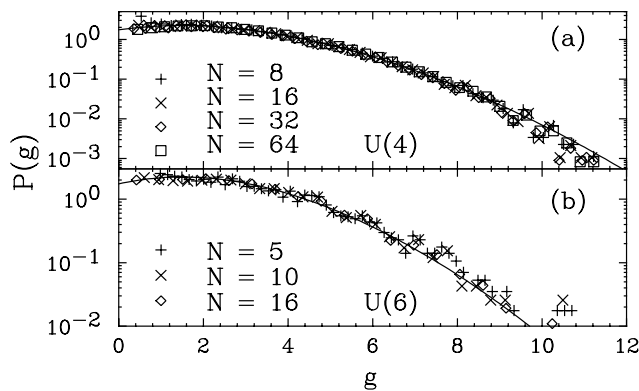


FIG. 4. (a) Distribution of energy gaps $p(g)$ where $g = \sqrt{2} \times N[E_{\min}(J=1) - E_{\min}(J=0)]$. The description is good, demonstrating a predicted scaling, with the exception of the behavior at the origin. (b) Same as (a) but for the nuclear case. Here $g = N[E_{\min}(J=2) - E_{\min}(J=0)]$.

Certainly the functional dependence of these moments are distinct from the dependence of (5) and (6), which is related to the ground state problem. Hence, the level densities do not reflect ground state properties in the sense that correlations between subspaces of different J , which are central to that question, are not reflected in these functions.

The distribution of $1^- - 0^+$ energy gaps, $\tilde{g} = E_{\min}^{1/N} - E_{\min}^0$, denoted $p(g)$, is obtained by averaging \tilde{g} over Eq. (4):

$$p(g) \propto \exp(-g^2/4r_2^2) [B_1 \exp(-g^2(r_1^2 + r_1 \cdot r_2)/\Delta) \times \text{erfc}(EB_2) - \text{erfc}(gB_3)], \quad (11)$$

where we find the scaling $g = \sqrt{2}N\tilde{g}$. Here, $B_1 = r_2/\sqrt{r_2^2 + 4r_1 \cdot r_2 + 4r_1^2} = 1.059$, $B_2 = (2r_1^2 + r_1 \cdot r_2 - \Delta)/2\sqrt{d\Delta} = -0.03$, $B_3 = (r_1 \cdot r_2 + r_2^2)/(2r_2\sqrt{d}) = -0.311$, $d = r_1^2r_2^2 - (r_1 \cdot r_2)^2$, and $\Delta = 4(r_1^2 + r_1 \cdot r_2) + r_2^2$ (the numerical value is for $N = 64$). This scaling in N is evident in Fig. 4(a) when we plot (11) versus g . We note that $p(g)$ describes the shape away from the origin. [Near the origin there is an abundance of small gaps that arise from the omitted region $z \in (0, 1)$]. We use the same function in Fig. 4(b) to compare to the $E_{2^+} - E_{0^+}$ gaps computed in the nuclear TBRE, with $g = N\tilde{g}$. The same scaling is apparent. Finally, the distribution of $R_{4/2} = (E_{4^+} - E_{0^+})/(E_{2^+} - E_{0^+})$ has been measured in the U(6) TBRE. The analogous quantity in this model is $R_{2/1} = (E_{2^+} - E_{0^+})/(E_{1^-} - E_{0^+})$. The distribution $P(R_{2/1})$ can be derived and has peaks at 2 and 3 [= $(J+1)/(J-1)$ for $J=2$], consistent with those in the U(6) model. To leading order this function is just a sum of two delta functions, and higher order corrections must be included to get the shape.

We have examined a bosonic TBRE which shares the salient features of more complex TBREs, such as a Wigner

limit for small N , Gaussian level densities for certain states, preponderance of 0^+ ground states, and so forth. By mapping the TBRE onto random polynomials on the unit interval, we are able to analytically understand many properties of the TBRE found numerically, including the frequency of 0^+ ground states. We find the latter is not attributed to the width of the level densities or the dimension of the Hilbert spaces. Rather, the various interactions in H tend to put the extreme values of spin ($J=0, N$) at the ends of the spectra, enhancing their chances to be the ground state. These results provide the first analytic understanding of ground state properties, distributions of lowest eigenvalues, gaps and level densities for the TBRE. These functions are also found to describe the nuclear properties obtained in the IBM. Scaling behavior has also been predicted and verified in certain observables. Since the analytic Lanczos approach can be generally applied, it would be interesting to see if a more general connection can be made between the TBREs and random polynomials [12].

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