No Black-Hole Theorem in Three-Dimensional Gravity

Daisuke Ida*
Department of Physics, Kyoto University, Kyoto 606-8502, Japan
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A common property of known black-hole solutions in (2 + 1)-dimensional gravity is that they require a negative cosmological constant. To explain this, it is shown in this Letter that a (2 + 1)-dimensional gravity theory which satisfies the dominant energy condition forbids the existence of a black hole.

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The (2 + 1)-dimensional theory provides us with one useful approach to more complicated (3 + 1)-dimensional classical gravity or conceptual problems in quantum gravity [1]. At first sight, the (2 + 1)-dimensional gravity looks trivial. In particular, the vacuum Einstein equation implies that the space-time is locally flat, corresponding to the absence of the gravitational radiation (Weyl tensor) in three dimensions. However, the local distribution of matter fields has a global effect on the outer empty space; for instance, the gravitational field of a point particle is described by a conical space with its deficit angle corresponding to the mass of the particle [2], which causes the gravitational lens effect. One should also note that the triviality of local geometry does not necessarily imply the triviality of the theory itself; namely, the topological degrees of freedom plays an important role in the theory of gravitation [3,4]. The triviality of local geometry in the (2 + 1)-gravity theory holds even if the cosmological term is taken into account. The Einstein space is simply a space of constant curvature in three dimensions, so that educated relativists would not imagine that there was a black-hole solution in this theory until in 1992 when Bañados *et al.* showed that there actually exists a black hole in the locally anti–de Sitter space [5,6]. This black-hole space-time, called BTZ black hole, is obtained by identifying certain points of (the covering manifold of) the anti–de Sitter space. A different identification makes a space-time representing the BTZ black hole in a closed universe [7], multiple BTZ black holes [8], or the creation of the BTZ black hole [9]. The BTZ black hole is characterized by the mass, angular momentum, and cosmological constant and has almost all the features of the Kerr-anti–de Sitter black hole in the conventional four-dimensional Einstein gravity. The BTZ black hole was shown to be also the solution of a low energy string theory [10,11].

Since the discovery of the BTZ black hole, a number of authors have attempted to find a black-hole solution in various theories in (2 + 1) dimensions. Black holes in topologically massive gravity [12] with the negative cosmological constant were found by Nutku [13]. In the Einstein-Maxwell-Λ system, a static (nonrotating) charged black hole had already been noted in the original paper by Bañados *et al.* [5]. Clément [14] generated from the charged BTZ black hole a class of rotating charged black holes. Though rotating solutions in Einstein-Maxwell-Λ theory seem to have infinite total mass and angular momentum [15], these divergences may be cured by adding a Chern-Simons term to the action [14]. Black holes with a dilaton field have been discussed by many authors. In Brans-Dicke theory, Sa *et al.* found black-hole solutions [16,17], and their properties were extensively studied for different Brans-Dicke parameters. Black holes in Einstein-Maxwell-dilaton-Λ theory were obtained by Chan and Mann in nonrotating [18] and rotating [19] cases. Other families were given by Koikawa *et al.* [20] and by Fernando [21]. Chen [22] also derived rotating black-hole solutions in this theory by means of the duality transformation in the equivalent nonlinear σ model. Black holes coupled to a topological matter field [23], conformal scalar field [24], Yang-Mills field [25], Born-Infeld field [26], etc. were also discussed.

Thus, many black-hole solutions are known. Here, it might be interesting to note that all the black-hole solutions listed above require a negative cosmological constant; otherwise a certain kind of energy condition is violated. A typical example might be the BTZ black hole. As already mentioned, the BTZ black hole may be constructed by making identifications in the anti–de Sitter space. We may also consider a similar construction in the de Sitter space. In this case, a natural procedure might be identifying two geodesic circles in each Poincaré disk associated with the open chart of the de Sitter space. The resultant space-time represents an inflating universe rather than a black hole. The absence of black hole in this example might be due to the difference in the causal structure of conformal infinity [27].

The purpose of this Letter is to give a reason for this situation. In particular, we will be able to answer the question: *"Why does the BTZ black hole require a negative cosmological constant?"* In the following, we consider the possibility of the existence of a black hole (in the sense of an apparent horizon) in three-dimensional space-time with the procedure given by Hawking [28] in terms of the spin-coefficient formalism [29].

Let (M, g) be a three-dimensional space-time and let Σ be a spacelike hypersurface in M. Suppose that Σ contains outer trapped surfaces; then there will be an apparent

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horizon H which is defined to be the outer boundary of the trapped region in Σ, where the notion "outer" is assumed to be well defined as in the case of the asymptotically flat (or anti–de Sitter) space-time. We also assume that the apparent horizon H is a smooth closed curve in Σ. Let m be a unit tangent vector of H, and let n and n' be future directed outgoing and ingoing null vectors orthogonal to H, respectively, such that $g(n, n') = 1$. The vectors n and n' are arranged such that $n - n'$ lies in Σ, which is always possible by means of the boost transformation $n \mapsto a^2 n$, $n' \mapsto a^{-2} n'$ by some positive function a. Let us consider a local deformation of H within Σ outside the trapped region generated by a vector field $X = e^f(n - n')$ with some smooth function f. Accordingly, the null triad $\{n, n', m\}$ is extended such that the normalization $g(n, n') = -g(m, m) = 1$, $g(n, n) = g(n', n') = g(n, m) = g(n', m) = 0$ is preserved and that m is tangent to each deformed H. Then, since X and $Y = e^h m$ form holonomic base vectors on Σ for some function h, n and n' are propagated such that

$$\delta f = \kappa - \tau + \beta = \kappa' - \tau' - \beta, \tag{1}$$

where Ricci rotation coefficients

$$\kappa = g(m, Dn), \qquad \tau = g(m, D'n), \qquad \beta = g(n', \delta n), \\ \kappa' = g(m, D'n'), \qquad \tau' = g(m, Dn'), \tag{2}$$

and the differential operators

$$D = \nabla_n, \qquad D' = \nabla_{n'}, \qquad \delta = \nabla_m \tag{3}$$

are defined following the spin-coefficient formalism in four space-time dimensions [29]. The convergence of light rays emitted outward from each deformed H is measured by the quantity

$$\rho = g(m, \delta n). \tag{4}$$

In particular, $\rho = 0$ holds on H since H will be a marginally trapped surface. The change in ρ along X is derived by the following equations:

$$D\rho - \delta\kappa = (\epsilon + \rho)\rho - (2\beta + \tau + \tau')\kappa + \phi_{++}, \tag{5}$$

$$D'\rho - \delta\tau = -\epsilon'\rho - \kappa\kappa' - \tau^2 + \rho\rho' - \phi_{+-} - \Pi, \tag{6}$$

where

$$\epsilon = g(n', Dn), \qquad \epsilon' = g(n, D'n'), \qquad \rho' = g(m, \delta n'), \\ \phi_{++} = \phi(n, n), \qquad \phi_{+-} = \phi(n, n'), \qquad \Pi = R/6 \tag{7}$$

with the trace-free part of the Ricci tensor $\phi = -\mathrm{Ric} + (R/3)g$. Subtracting Eq. (6) from Eq. (5), we obtain the equation

$$\begin{aligned} e^{-f}\mathcal{L}_X\rho &= \delta(\kappa - \tau) - (2\beta + \tau + \tau')\kappa + \kappa\kappa' + \tau^2 \\ &\quad + \phi_{++} + \phi_{+-} + \Pi \\ &= \delta(\delta f - \beta) + (\kappa - \tau)^2 + \phi_{++} \\ &\quad + \phi_{+-} + \Pi \end{aligned} \tag{8}$$

on H, where Eq. (1) has been used. Now suppose that there is a positive cosmological constant $\Lambda > 0$ and that the stress-energy tensor T satisfies *the dominant energy condition:* (i) $T(W, W) \geq 0$ *and* (ii) $T(W)$ *is nonspacelike, for every timelike vector* W. Then, the Einstein equation $\mathrm{Ric} - (R/2)g + \Lambda g = -8\pi T$ leads to the inequalities

$$\phi_{++} \geq 0, \qquad \phi_{+-} + \Pi > 0. \tag{9}$$

The term $\delta(\delta f - \beta)$ in the last line of Eq. (8) can be made zero by appropriately choosing the function f; in fact, parametrizing H by the proper length $s \in [0, \mathrm{length}(H))$, such a function f can be explicitly written as

$$f = \int^s \beta\, ds - \left(\frac{\oint \beta\, ds}{\oint ds}\right) s. \tag{10}$$

Then, the last line of Eq. (8) is positive definite, $\mathcal{L}_X\rho > 0$. This implies that there is an outer trapped surface outside H, which contradicts the assumption that H is the outer boundary of such surfaces. Hence, we obtain the following no black-hole theorem.

Theorem 1: *Let* (M, g) *be a three-dimensional space-time subject to the Einstein equation* $\mathrm{Ric} - (R/2)g + \Lambda g = -8\pi T$ *with* $\Lambda > 0$. *If the stress-energy tensor* T *satisfies the dominant energy condition, then* (M, g) *contains no apparent horizons.*

This explains why black-hole solutions require a negative cosmological constant. Strictly speaking, we can say only that there is no nondegenerate apparent horizon ($\rho = 0$, $\mathcal{L}_X\rho \neq 0$) in the case of $\Lambda = 0$; however, the presence of matter fields such as the dilaton or Maxwell field will exclude even degenerate horizons.

Thus, a black hole in (2 + 1) gravity requires negative energy such as a negative cosmological constant. This implies the breakdown of the predictability in certain three-dimensional theories. As in four space-time dimensions, we may consider the Oppenheimer-Snyder model of the gravitational collapse. The homogeneous disk of dust will collapse to a central point and a naked conical singularity will be left. This picture of gravitational collapses will remain unchanged unless the negative cosmological constant is added. Even in the case of the nonsymmetric gravitational collapse of gauge fields or scalar fields, there will not form a black hole, so that when a singularity is formed such a singularity will be naked.

We have discussed the existence problem of apparent horizons, while the black hole is often defined by the event horizon. Since Theorem 1 relies on the local analysis, we cannot argue the global structure of space-time such as an event horizon. An important exception is the stationary case; we can replace "apparent horizons" with "stationary event horizons" in Theorem 1, since it is known that these coincide in this case.

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*Electronic address: ida@tap.scphys.kyoto-u.ac.jp

[1] S. Carlip, *Quantum Gravity in (2 + 1)-Dimensions* (Cambridge University Press, Cambridge, England, 1998).
[2] S. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (N.Y.) **152**, 220 (1984).
[3] E. Witten, Nucl. Phys. **B311**, 46 (1988).
[4] A. Hosoya and K. Nakao, Classical Quantum Gravity **7**, 163 (1990).
[5] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[6] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[7] M. Siino, Classical Quantum Gravity **11**, 1995 (1994).
[8] D. R. Brill, Phys. Rev. D **53**, R4133 (1996).
[9] H-J. Matschull, Classical Quantum Gravity **16**, 1069 (1999).
[10] G. T. Horowitz and D. L. Welch, Phys. Rev. Lett. **71**, 328 (1993).
[11] N. Kaloper, Phys. Rev. D **48**, 2598 (1993).
[12] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[13] Y. Nutku, Classical Quantum Gravity **10**, 2657 (1993).
[14] G. Clément, Phys. Lett. B **367**, 70 (1996).
[15] M. Cataldo and P. Salgado, Phys. Lett. B **448**, 20 (1999).
[16] P. M. Sa, A. Klebor, and J. P. S. Lemos, Classical Quantum Gravity **13**, 125 (1996).
[17] P. M. Sa and J. P. S. Lemos, Phys. Lett. B **423**, 49 (1998).
[18] K. C. K. Chan and R. B. Mann, Phys. Rev. D **50**, 6385 (1994).
[19] K. C. K. Chan and R. B. Mann, Phys. Lett. B **371**, 199 (1996).
[20] T. Koikawa, T. Maki, and A. Nakamula, Phys. Lett. B **414**, 45 (1997).
[21] S. Fernando, Phys. Lett. B **468**, 201 (1999).
[22] C. M. Chen, Nucl. Phys. **B544**, 775 (1999).
[23] S. Carlip, J. Gegenberg, and R. B. Mann, Phys. Rev. D **51**, 6854 (1995).
[24] C. Martínez and J. Zanelli, Phys. Rev. D **54**, 3830 (1996).
[25] V. Brindejone, Classical Quantum Gravity **15**, 3599 (1998).
[26] M. Cataldo and A. García, Phys. Lett. B **456**, 28 (1999).
[27] M. Siino (private communication).
[28] S. W. Hawking, Commun. Math. Phys. **25**, 152 (1972); in *Black Holes,* edited by C. De Witt and B. S. De Witt (Gordon and Breach, New York, 1973).
[29] R. Penrose and W. Rindler, *Spinors and Space-Time* (Cambridge University Press, Cambridge, England, 1986), Vol. 1.