Dimension of Fractal Growth Patterns as a Dynamical Exponent

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We consider a conformal theory of fractal growth patterns in two dimensions, including diffusion limited aggregation (DLA) as a particular case. In this theory the fractal dimension of the asymptotic cluster manifests itself as a *dynamical* exponent observable already at very early growth stages. Using a renormalization relation we show from early stage dynamics that the dimension D of DLA can be estimated, $1.69 < D < 1.72$. We explain why traditional numerical estimates converged so slowly. We discuss similar computations for other fractal growth processes in two dimensions.

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The diffusion limited aggregation (DLA) model was introduced in 1981 by Witten and Sander [1]. The model is important as an example of fractal pattern formation under very simple rules; it was shown to underlie many pattern forming processes including dielectric breakdown [2], viscous fingering [3], electrochemical deposition [4], and bacterial growth [5]. The algorithm begins with fixing one particle at the center of coordinates in d dimensions, and follows the creation of a cluster by releasing random walkers from infinity, allowing them to walk around until they hit any particle belonging to the cluster. The fundamental difficulty of this and similar growth processes is that their mathematical description calls for solving equations with boundary conditions on a complex, evolving interface. In addition, the growth probability for a random walker to hit the interface (known as the "harmonic measure") has been shown to be a multifractal measure [6] characterized by infinitely many exponents [7,8]. Until now these difficulties defied all attempts to understand the fractal properties of DLA. In fact, even the numerical estimates [9] of the fractal dimension D of DLA clusters turned out to converge extremely slowly with the number of particles n of the cluster, leading even to speculations [10] that for $n \to \infty$ the clusters were plane filling (i.e., $D = 2$). In this Letter we employ a conformal theory of fractal growth processes in two dimensions, including DLA as a special case. We show how to compute the dimension from early stage dynamics, and explain why in the old techniques convergence was so miserable. We introduce the novel notion that the fractal dimension D appears as a *dynamical* exponent already at early stages of the growth in which the cluster has not yet developed a fractal geometry. This exponent is apparent when there are only 1–2 layers of growing particles, much before it has the geometric meaning of dimension.

For continuous time processes in two dimensions the above-mentioned difficulties were efficiently dealt with in the past [11,12] by considering the conformal map from the unit circle to the advancing interface. This way the "interface" in the mathematical plane remains forever simple, and the complexity of the evolving interface is delegated to the dynamics of the conformal map. For discrete particle growth such a language was developed recently [13–16], showing that a variety of fractals in two dimensions can be grown by iterating conformal maps.

Once a fractal object is well developed, it is extremely difficult to find a conformal map from a smooth region to its boundary, simply because the conformal map is terribly singular on the tips of a fractal shape. Rather, in recent work it was shown how to grow the cluster by iteratively constructing the conformal map [13–16]. We briefly summarize the main points of this approach. Consider $\Phi^{(n)}(w)$ which conformally maps the exterior of the unit circle $e^{i\theta}$ in the mathematical w plane onto the complement of the (simply connected) cluster of n particles in the physical z plane. The unit circle is mapped to the boundary of the cluster. The map $\Phi^{(n)}(w)$ is made from compositions of elementary maps $\phi_{\lambda,\theta}$,

$$\Phi^{(n)}(w) = \Phi^{(n-1)}(\phi_{\lambda_n,\theta_n}(w)), \qquad (1)$$

where the elementary map $\phi_{\lambda,\theta}$ transforms the unit circle to a circle with a "bump" of linear size $\sqrt{\lambda}$ around the point $w = e^{i\theta}$. An example of a good elementary map $\phi_{\lambda,\theta}$ was proposed in [13], endowed with a parameter a in the range $0 < a < 1$, determining the shape of the bump. We employ here $a = 2/3$. Accordingly the map $\Phi^{(n)}(w)$ adds on a new bump to the image of the unit circle under $\Phi^{(n-1)}(w)$. The bumps in the z plane simulate the accreted particles in the physical space formulation of the growth process. Since we want to have *fixed size* bumps in the physical space, say, of fixed area λ_0, we choose in the nth step $\lambda_n = \lambda_0/|\Phi^{(n-1)'}(e^{i\theta_n})|^2$. The recursive dynamics can be represented as iterations of the map $\phi_{\lambda_n,\theta_n}(w)$: $\Phi^{(n)}(w) = \phi_{\lambda_1,\theta_1} \circ \phi_{\lambda_2,\theta_2} \circ \cdots \circ \phi_{\lambda_n,\theta_n}(\omega)$.

The difference between various growth models will manifest itself in the different itineraries $\{\theta_1 \cdots \theta_n\}$. To grow a DLA we have to choose random positions θ_n. This way we accrete fixed size bumps in the physical plane according to the harmonic measure (which is transformed into a uniform measure by the analytic inverse of $\Phi^{(n)}$). The DLA cluster is fully determined by the stochastic itinerary $\{\theta_k\}_{k=1}^{n}$. In Fig. 1 we present a typical DLA cluster grown by this method to size $n = 10^5$. Other fractal clusters can be obtained by choosing a nonrandom itinerary [16]. A beautiful family of growth patterns is obtained

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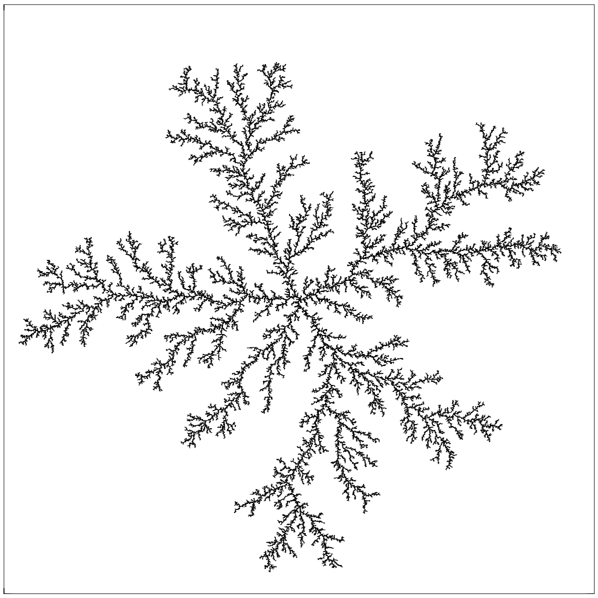

FIG. 1. A DLA cluster, $n = 100\,000$.

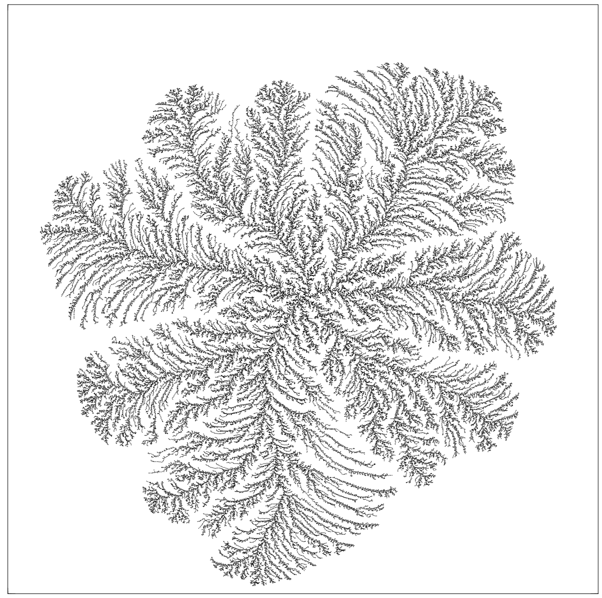

FIG. 2. A deterministic cluster with a golden mean itinerary, $n = 100\,000$.

from quasiperiodic itineraries: $\theta_{k+1} = \theta_k + 2\pi W$, where W is a quadratic irrational number. An example is shown in Fig. 2, in which W is the golden mean $(\sqrt{5} + 1)/2$. In [16] it was argued that itineraries obtained by using other values of quadratic irrationals for W lead to clusters of different appearance, but the same dimension, which was estimated numerically to be $D = 1.86 \pm 0.03$. In the same paper other deterministic itineraries (not obtained from circle maps) were shown to lead to clusters with different dimensions. One (trivial) example that is nevertheless useful for our consideration below is the itinerary $\theta_k = 0$ for all k. Such an itinerary grows a one-dimensional wire of width $\sqrt{\lambda_0}$.

As stressed in [13–16] the advantage of conformal maps is that they afford us analytic power that is not obtainable otherwise. To understand this consider the Laurent expansion of $\Phi^{(n)}(w)$:

$$\Phi^{(n)}(w) = F_1^{(n)} w + F_0^{(n)} + F_{-1}^{(n)} w^{-1} + F_{-2}^{(n)} w^{-2} + \ldots . \tag{2}$$

The recursion equations for the Laurent coefficients of $\Phi^{(n)}(w)$ can be obtained analytically, and, in particular, one shows that [13,14] $F_1^{(n)} = \prod_{k=1}^{n}[1 + \lambda_k]^a$. The first Laurent coefficient $F_1^{(n)}$ determines the fractal dimension of the cluster, being identical to the Laplace radius which is the radius of a charged disk having the same field far away as the charged cluster [14]. Moreover, defining R_n as the minimal radius of all circles in z that contain the n cluster, one can prove that [17] $R_n \leq 4F_1^{(n)}$. Accordingly one expects that for sufficiently large clusters (to be made precise below)

$$F_1^{(n)} \sim n^{1/D}\sqrt{\lambda_0}, \tag{3}$$

as $\sqrt{\lambda_0}$ remains the only length scale in the problem when the radius of the cluster is much larger than the radius of the initial smooth interface (which we take as the unit circle in this discussion).

These observations lead now to the central development of this Letter, and to the most important result. Consider a renormalization process in which we fix the initial smooth interface, but change λ_0, and then rescale n such as to get the "same" cluster. Of course, we need to specify what do we mean by the "same" cluster, and a natural requirement is that the electrostatic field on coarse scales (i.e., far from the cluster) will remain *invariant*. In other words, we should require the invariance of the Laplace radius $F_1^{(n)}$ (and possibly of additional low order Laurent coefficients) under renormalization. Clearly, for a given itinerary $\{\theta_k\}_{k=1}^{n}$, $F_1^{(n)}$ is a function of n and λ_0 only. Accordingly, considering Eq. (3), we note that a renormalization process can reach a fixed point if and only if $F_1^{(n)}(\lambda_0)$ attains a nontrivial fixed point function F_1^* of the single "scaling" variable $x = \sqrt{\lambda_0}\, n^{1/D}$. Obviously in the asymptotic limit $x \gg 1$, $F_1^{(n)}(\lambda_0)$ must converge to F_1^* which is *linear* in x in this regime. The main new findings of this Letter are that F_1^* exists as a nonlinear function of x, and that $F_1^{(n)}(\lambda_0)$ converges (within every universality class) to its fixed point function F_1^* already for $x \ll 1$.

In principle one can demonstrate the convergence to F_1^* analytically. This is easy to do in the case of the degenerate itinerary growing a wire. In this case convergence is achieved after the addition of 2–3 bumps, even in the limit $\lambda_0 \to 0$; see Fig. 3. For $x \to 0$ the function $F_1^*(x) \sim x^2$. For other nontrivial itineraries it becomes increasingly cumbersome to demonstrate the convergence by hand. With the assistance of the machine we can demonstrate the convergence in all the other cases. In Fig. 4 we present $F_1^{(n)}(\lambda_0) - 1$ as a function of x for a

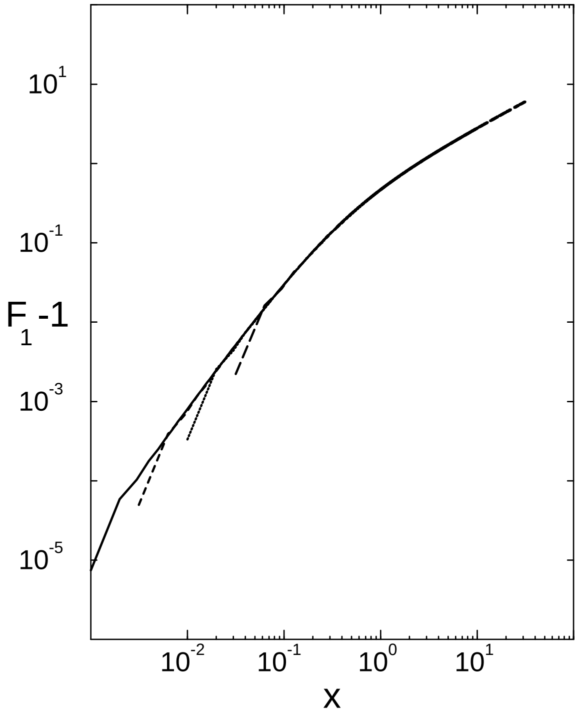

FIG. 3. The one-dimensional wire. We plot $F_1(x) - 1$ vs $x = n\lambda_0^{1/2}$ and demonstrate the convergence to the asymptotic nonlinear function F_1^*. The values of λ_0 used are 10^{-3}, 10^{-4}, 10^{-5}, and 10^{-6}.

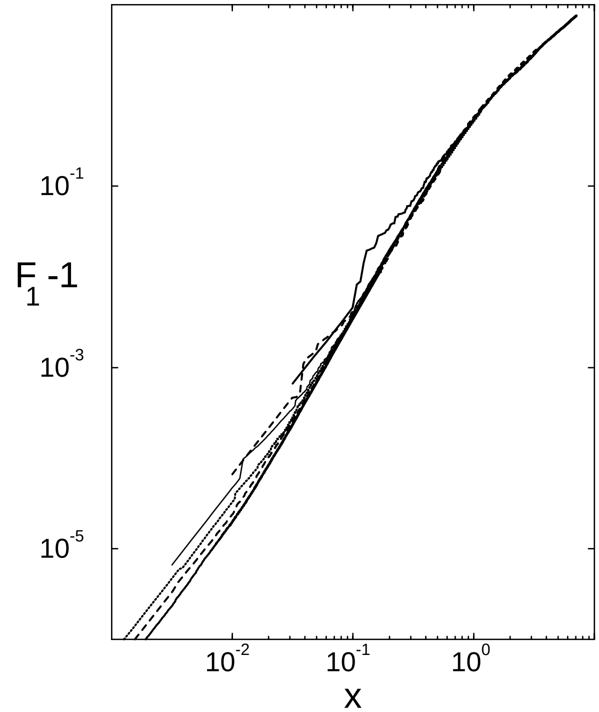

FIG. 4. Convergence to F_1^* for the DLA. We plot $F_1(x) - 1$ vs $x = n^{1/D}\lambda_0^{1/2}$ with $D = 1.7$. The values of λ_0 used are 10^{-3} (upper solid line), 10^{-4} (upper dashed), 10^{-5} (thin solid), 10^{-6} (dotted), 10^{-7} (lower solid), and 10^{-8} (lower dashed). The fixed point function F_1^* is best approximated by the lowest curve in the figure. Convergence to the asymptotic nonlinear function F_1^* is seen for smaller x values when λ_0 decreases.

typical DLA itinerary and for values of λ_0 ranging between 10^{-8} to 10^{-3}. We note that for $\lambda_0 \to 0$ the convergence to the fixed point function is obtained infinitesimally close to the initial circle for which $F_1^{(n=0)} = 1$. In fact, convergence for this itinerary, as well as for all other nontrivial itineraries, is obtained for $n \geq n_c$ where $n_c \approx 2\pi/\sqrt{\lambda_0}$. This is the number of bumps required to obtain one-layer coverage of the original circular interface. Obviously $n_c^{1/D}\sqrt{\lambda_0} \to 0$ for $\lambda_0 \to 0$, demonstrating the convergence to F_1^* for $x \ll 1$. In Fig. 5 we exhibit the convergence for the golden mean itinerary. Note that the fixed point functions are different, and they both differ from the wire case. The main point of this analysis is that convergence to F_1^* can be obtained for x arbitrarily small by going to the limit $\lambda_0 \to 0$.

The existence of a fixed point function translates immediately to a calculational scheme. Consider a given itinerary $\{\theta_k\}_{k=1}^N$ of one of the above classes, and calculate $F_1^{(n)}(\lambda_0)$ for $N > n > n_c(\lambda_0)$. Rescale now $\lambda_0 \to \lambda_0/s$, and calculate $F_1^{(n')}(\lambda_0/s)$ for $N > n' > n_c(\lambda_0/s)$. We can compute D from finding the value n' which preserves the Laplace radius under rescaling of λ_0 by s: $(n'/n)^{1/D} = \sqrt{s}$. Since F_1^* is monotonic, there is only one solution:

$$D = \frac{2[\log n' - \log n]}{\log s}. \tag{4}$$

As a first example consider the wire case. Computing $F_1^{(10)}(10^{-6})$ we find that for $\lambda_0 = 10^{-5}$ the same value of $F_1^{(n)}$ is obtained for n between 3 and 4. Equation (4) with $s = 10$ then predicts $0.796 < D < 1.045$. Repeating for $F_1^{(100)}(10^{-6})$ we find the same value of $F_1^{(n)}(10^{-5})$ for n between 31 and 32. From Eq. (4) $0.9897 < D < 1.0173$. We stress that this precision is obtained when $F_1^{(n)} - 1$ is still 0.0133. In this simple case any desired accuracy in extracting the dimension of the asymptotic cluster ($D = 1$) can be achieved here by decreasing λ_0 without increasing n. Second we consider the deterministic itinerary with golden mean winding number W. Using values of $F_1^{(n)}(10^{-6})$ and $F_1^{(n)}(10^{-5})$ between 1.10 and 1.20 we can bound the dimension of the cluster to $1.8305 < D < 1.8380$. Note that in this case we need to have at least one layer covering which is obtained only when $n_c \approx 2\pi/\sqrt{\lambda_0}$. Finally, we consider DLA. Here the itineraries are stochastic and one could imagine that only under extensive ensemble averaging one would obtain tight bounds on D. In fact, we find that using values of $F_1^{(n)}(10^{-8})$ and $F_1^{(n)}(2 \times 10^{-8})$ between 1.002 and 1.01 we can bound D as tightly as $1.69 < D < 1.72$. Note that to achieve this accuracy we did not need to go to high values of $F_1^{(n)}$, but rather used small values of λ_0 to reach convergence very early. This demonstrates again the unexpected fact that the *asymptotic* dimension appears as a renormalization

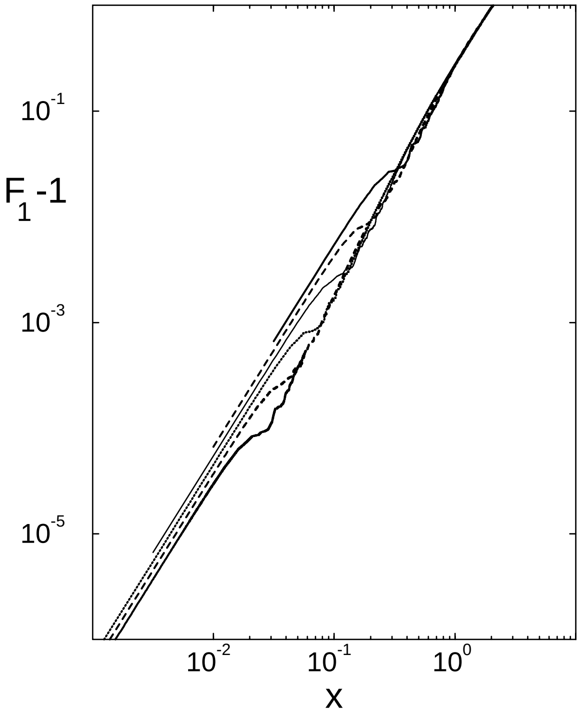

FIG. 5. Convergence to F_1^* for the golden mean itinerary. Data are the same as in Fig. 4 but $x = n^{1/D}\lambda_0^{1/2}$ with $D = 1.83$. The fixed point function F_1^* is best approximated by the lowest curves for $x > 3 \times 10^{-2}$ where the data with $\lambda_0 = 10^{-8}$ converges onto F_1^*.

exponent right after one or a few layers of particles cover the circle, and very much before $F_1^{(n)} \sim n^{1/D}$.

At this point we need to address two questions: (i) Why do classical numerical estimates [9,10] of the fractal dimension of DLA converge so slowly? In standard numerical experiments the radius of gyration of the grown cluster was plotted in log-log coordinates against the number of particles, with D estimated from the slope. Examining our fixed point functions F_1^* (see Figs. 3 and 4) we note the slow crossover to linear behavior, which is not fully achieved even for extremely high values of n. Thus reliable estimates of D from radius of gyration calculations require inhuman effort, as was indeed experienced by workers in the field [10]. In the present formulation the appearance of the *asymptotic* D as a renormalization exponent already at early stages of the growth allows a convergent calculation. (ii) Can the fractal dimension be determined without the multifractal spectrum? The multiscaling properties of the harmonic measure have left the impression that computing the fractal dimension of DLA requires a simultaneous control of the host of exponents characterizing the measure. The scaling relation $D_3 = D/2$ [18] strengthened this impression. The approach presented here indicates that $1/D$ appears in the dynamics much before the measure becomes multiscaling. A calculation of D from a few particle dynamics is presented in [19].

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