Dynamic Symmetries at the Critical Point

F. Iachello
Center for Theoretical Physics, Sloane Laboratory, Yale University, New Haven, Connecticut 06520-8120
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A new class of dynamic symmetries is introduced. It is suggested that an element of this class, associated with zeros of Bessel functions, be used to describe spectra of nuclei at or around the critical point of the U(5)-SO(6) shape phase transition, and, in general, spectra of systems undergoing a (second order) phase transition between the algebraic structures U($n-1$) and SO(n).

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Dynamic symmetries have provided a useful tool to describe properties of several physical systems. In the common definition [1], a dynamic symmetry is that situation in which the Hamiltonian operator, H, can be written in terms of Casimir operators C_i of a chain of algebras $G \supset G' \supset G'' \supset \ldots$. The most notable examples are the dynamic symmetries of the interacting boson model [2] in nuclear physics and those of the vibron model [3] in molecular physics. Dynamic symmetries of this type can be easily recognized by analyzing the algebraic structure of the problem. By breaking the algebra G into all its subalgebra chains, one can find all possible dynamic symmetries of G. In the interacting boson model, for example, where $G \equiv$ U(6), there are three possible dynamic symmetries usually labeled by the first subalgebra U(5), SU(3), SO(6) appearing in the chain. Dynamic symmetries are related to exactly solvable problems and produce all results for observables in explicit analytic form. As such they are extremely useful in the analysis of experimental data and have led to major discoveries [4].

In this Letter, I want to point out that there are other classes of dynamic symmetries that could be useful in the analysis of experimental data and which are also related to exactly solvable problems. These new symmetries describe systems undergoing phase transitions between the dynamical symmetries of the algebraic structure G and therefore extend the concept of dynamic symmetry to the most challenging and sensitive situation one may encounter in quantal systems.

The symmetries I wish to introduce cannot be recognized easily by looking at the algebra G, but rather result from consideration of differential equations

$$D\psi = E\psi\,, \tag{1}$$

where D is the differential operator representing the Hamiltonian H. In order to make things concrete, I construct here explicitly one of these cases, that has applications to the study of shape phase transitions in nuclei. Consider the Bohr Hamiltonian [5]

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_\kappa \frac{Q_\kappa^2}{\sin^2(\gamma - \frac{2}{3}\pi\kappa)}\right] + V(\beta,\gamma)\,. \tag{2}$$

This Hamiltonian lives in a five-dimensional space with two intrinsic variables β, γ and three Euler angles $\theta_i (i = 1,2,3)$. When the potential depends only on β, $V(\beta,\gamma) = U(\beta)$, by writing

$$\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi(\gamma,\theta_i) \tag{3}$$

one can separate variables in the standard way [6]

$$\begin{gathered}\left[-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} + \frac{1}{4}\sum_\kappa\frac{Q_\kappa^2}{\sin^2(\gamma-\frac{2}{3}\pi\kappa)}\right]\Phi(\gamma,\theta_i) = \Lambda\Phi(\gamma,\theta_i)\,,\\ \Lambda = \tau(\tau+3); \qquad \tau = 0,1,2,\ldots,\\ \left[-\frac{\hbar^2}{2B}\left(\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} - \frac{\Lambda}{\beta^2}\right) + U(\beta)\right]f(\beta) = Ef(\beta)\,.\end{gathered} \tag{4}$$

Introducing reduced energies and potentials $\varepsilon = \frac{2B}{\hbar^2}E$, $u = \frac{2B}{\hbar^2}U$, one can rewrite the equation in the β variable as

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{\Lambda}{\beta^2} + u(\beta)\right]f(\beta) = \varepsilon f(\beta)\,. \tag{5}$$

By setting $\varphi(\beta) = \beta^{3/2}f(\beta)$ one obtains

$$\varphi'' + \frac{\varphi'}{\beta} + \left[\varepsilon - u(\beta) - \frac{(\tau+3/2)^2}{\beta^2}\right]\varphi = 0\,. \tag{6}$$

The differential equation (6) possesses several dynamical symmetries of the type suggested here. As an example, I now discuss a case applicable to the study of nuclear spectra at the critical point of the U(5)-SO(6) [vibrator to γ-unstable rotor] shape phase transition. To this end, consider the case in which the potential is a five-dimensional infinite well

$$\begin{aligned} u(\beta) &= 0, \qquad \beta \le \beta_w\,,\\ u(\beta) &= \infty, \qquad \beta > \beta_w\,. \end{aligned} \tag{7}$$

In this case one obtains a Bessel equation with

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$\nu = \tau + 3/2$

$$\varphi'' + \frac{\varphi'}{z} + \left[1 - \frac{(\tau + 3/2)^2}{z^2}\right]\varphi = 0, \qquad z = \beta k, \tag{8}$$

with $k = \varepsilon^{1/2}$. The boundary condition $\varphi(\beta_w) = 0$ determines the eigenvalues to be

$$E_{\xi,\tau} = \frac{\hbar^2}{2B} k_{\xi,\tau}^2, \qquad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_w}, \tag{9}$$

where $x_{\xi,\tau}$ is the ξth zero of $J_{\tau+3/2}(z)$, and the eigenfunctions

$$\begin{aligned} \varphi_{\xi,\tau}(\beta) &= c_{\xi,\tau} J_{\tau+3/2}(k_{\xi,\tau}\beta), \\ f_{\xi,\tau}(\beta) &= c_{\xi,\tau} \beta^{-3/2} J_{\tau+3/2}(k_{\xi,\tau}\beta). \end{aligned} \tag{10}$$

The normalization constants $c_{\xi,\tau}$ can be obtained by imposing the condition

$$\int_0^\infty \beta^4 \, d\beta f^2(\beta) = 1. \tag{11}$$

This problem is thus exactly solvable. The corresponding symmetry will be denoted by E(5) since the Bessel functions form a basis for the representations of the Euclidean group [7], and five is the number of dimensions of the problem. In general, all problems for which the eigenvalues of H are given in terms of zeros of special functions form another class of exactly solvable problems and hence of dynamic symmetries, which can be called "representation symmetries," since they are related to the representations of some group $\tilde{G}$. The case in which the special function is the Bessel function is an element of this class. Other elements are formed, for example, by the zeros of the Airy functions, which appear for a potential $u(\beta) \propto \beta$.

Once one has the zeros of the special functions, one can calculate all observables. Rather than giving those values, I give in Table I the excitation energies of the lowest states for this symmetry, normalized to the energy of the first excited state.

It should be noted that the symmetry fixes uniquely the values of the energy eigenvalues. Only an overall scale is needed in comparing with experiment. The spectrum corresponding to Eq. (9) is shown in Fig. 1. For each τ, the values of the allowed angular momenta are obtained by the usual reduction SO(5) $\supset$ SO(3) [8], which is still a symmetry of the problem, and are given in Table IV of that reference. Particularly noteworthy from Table I are the ratios $E_{4_{1,2}}/E_{2_{1,1}} = 2.20, E_{2_{1,2}}/E_{2_{1,1}} = 2.20$ and $E_{0_{2,0}}/E_{2_{1,1}} = 3.03$, where the states are denoted by $L_{\xi,\tau}$.

TABLE 1. Excitation energies of the E(5) symmetry.

	$\xi = 1$	$\xi = 2$	$\xi = 3$	$\xi = 4$
$\tau = 0$	0	3.03	7.58	13.64
$\tau = 1$	1	4.80	10.11	16.93
$\tau = 2$	2.20	6.78	12.86	20.44
$\tau = 3$	3.59	8.97	15.81	24.16

Electromagnetic transition rates can be calculated by taking matrix elements of the transition operators. Particularly interesting are the matrix elements of the quadrupole operator

$$\begin{aligned} T^{(E2)} &= t\alpha_{2\mu} \\ \alpha_{2\mu} &= \beta\left[D_{\mu,0}^{(2)}\cos\gamma + \frac{1}{\sqrt{2}}(D_{\mu,2}^{(2)} + D_{\mu,-2}^{(2)})\sin\gamma\right], \end{aligned} \tag{12}$$

where t is a scale factor. The γ, θ_i part of the calculation can be done in the standard way. One is left with the β part

$$\int_0^\infty \beta f_{\xi,\tau}(\beta) f_{\xi',\tau'}(\beta)\beta^4 \, d\beta = I_{\xi,\tau;\xi',\tau'}. \tag{13}$$

Evaluation of these integrals, as well as of the γ, θ_i part, gives the results of Fig. 1. Here rather than the matrix elements, the $B(E2)$ values are shown. [$B(E2; L \to L') = \frac{|\langle L|T|L'\rangle|^2}{2L+1}$.] Again it should be noted that the symmetry fixes uniquely the values of the matrix elements. All $E2$ transition rates are given in terms of an overall scale. Of great interest are results for the ratios $R = \frac{B(E2;4_{1,2}\to 2_{1,1})}{B(E2;2_{1,1}\to 0_{1,0})} = 1.68$, $R' = \frac{B(E2;2_{1,2}\to 2_{1,1})}{B(E2;2_{1,1}\to 0_{1,0})} = 1.68$, $R'' = \frac{B(E2;0_{2,0}\to 2_{1,1})}{B(E2;2_{1,1}\to 0_{1,0})} = 0.86$.

The dynamic symmetries described here are of interest in all those situations where the potential has a flat behavior as a function of some coordinate. This situation appears typically when the system undergoes a phase transition. In the case discussed here the phase transition is in the coordinate β (shape phase transition) and corresponds to the U(5)-SO(6) transition of the algebraic structure U(6). The potentials just before, at, and just after the phase transition, obtained from the Hamiltonian of the interacting boson model by the method of intrinsic or coherent states [9–11] are shown in Fig. 2. This potential is given, for the group coherent states, by [12]

$$u(\beta) = \frac{1}{2}(1 - \eta)\beta^2 + \frac{\eta}{4}(1 - \beta^2)^2, \tag{14}$$

where η is the control parameter. At, and around the critical point $\eta = \frac{1}{2}$, the potential has a flat behavior. Hence the dynamic symmetry associated with the zeros of the Bessel functions is a good starting point to describe the experimental situation at the critical point of the U(5)-SO(6) shape phase transition in nuclei. Since several nuclei are in this situation (Xe, Ba, ...), the symmetry can be used for these cases, as will be discussed in the accompanying paper [13].

In general, the symmetries associated with the zeros of the Bessel functions can be used to describe the spectra of systems undergoing phase transitions of the type U($n - 1$)-SO(n). All these phase transitions should have a

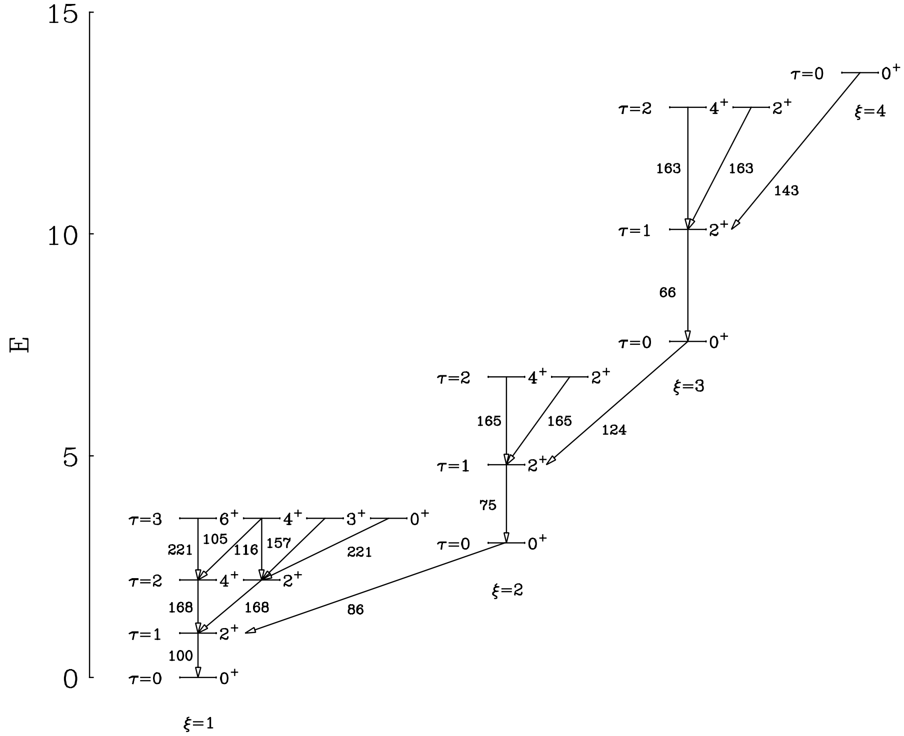

FIG. 1. Schematic representation of the lowest portion of the spectrum of the five-dimensional infinite well [E(5) symmetry]. Energies are in units of the energy of the first excited state, $E_{2_{1,1}}$. $B(E2)$ values are in units of $B(E2; 2_{1,1} \rightarrow 0_{1,0}) = 100$.

universal behavior in their spectra. The distinguishing features of these spectra are large and positive anharmonicities (as one can see in Fig. 1) and specific energy and intensities ratios.

In conclusion, I have suggested a new class of dynamic symmetries, associated with zeros of special functions, and shown explicitly how they can be used to describe properties of nuclei at the critical point of the U(5)-SO(6) shape phase transition. The symmetry described here can also be used for other second order transitions such as the U(3)-SO(4) transition which applies to van der Waals molecules [3]. This application will be presented elsewhere. Here I point out that this class of dynamic symmetry can be enlarged by considering potentials which are finite wells

$$\begin{aligned} u(\beta) &= 0, \qquad \beta \leq \beta_w , \\ u &= D, \qquad \beta > \beta_w . \end{aligned} \tag{15}$$

These problems are also exactly solvable, except that the solutions are given not by the zeros of special functions but by a transcendental equation involving those special functions at the matching point β_w. In the case of applications to nuclei, these other symmetries are particularly important and are related to the shape phase transitions for finite boson number N in the interacting boson model. A complete description of these symmetries, their relation with the infinite N case presented here and with the algebraic finite N solution of the interacting boson model for the transitional class U(5)-SO(6) given recently in Ref. [14], will be presented in a longer publication.

Another enlargement of the concept of dynamic symmetry introduced here is that of considering Hamiltonians

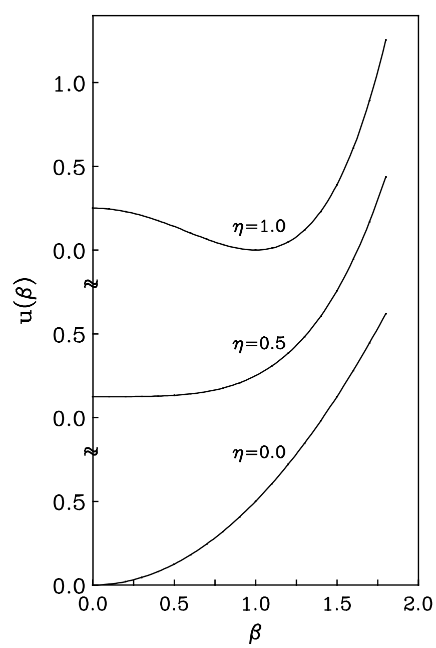

FIG. 2. Potential energy surfaces for the U(5)-SO(6) shape phase transitions obtained from the interacting boson model Hamiltonian by the method of "group" coherent states.

more general than Eq. (2), obtained by adding to it Casimir operators of the subalgebras SO(5) $\supset$ SO(3). For example by adding the Casimir operator of the subalgebra SO(3) one obtains the energy formula

$$E(\xi, \tau, \nu_\Delta, L, M_L) = E_0 + Ak^2_{\xi,\tau} + CL(L + 1), \quad (16)$$

to be compared with that of the SO(6) symmetry [8]

$$E(\sigma, \tau, \nu_\Delta, L, M_L) = E_0 + A\sigma(\sigma + 4) + B\tau(\tau + 3) + CL(L + 1). \quad (17)$$

One can observe that at the critical point the number of parameters is reduced by one, since the ξ and τ dependence is given by $k^2_{\xi,\tau}$.

Finally, symmetries based on zeros of special functions can also be used to discuss the U(5)-SU(3) shape phase transition in nuclei. However, this phase transition involves both variables β and γ simultaneously and its treatment is much more complex than the case discussed in this article.

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