Generic Isolated Horizons and Their Applications

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The notion of isolated horizons is extended to allow for distortion and rotation. Space-times containing a black hole, itself in equilibrium but possibly surrounded by radiation, satisfy these conditions. The framework has three types of applications: (i) it provides new tools to extract physics from *strong* field geometry; (ii) it leads to a generalization of the zeroth and first laws of black hole mechanics and sheds new light on the "origin" of the first law; and (iii) it serves as a point of departure for black hole entropy calculations in nonperturbative quantum gravity.

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A great deal of analytical work on black holes in general relativity centers around event horizons in globally stationary space-times (see, e.g., [1,2]). While it is a natural starting point, this idealization seems overly restrictive from a physical perspective. In a realistic gravitational collapse, or a black hole merger, the final black hole is expected to rapidly reach equilibrium. However, the exterior space-time region will not be stationary. Indeed, a primary goal of many numerical simulations is to study radiation emitted in the process. Similarly, since event horizons can be determined only retroactively after knowing the entire space-time evolution, they are not directly useful in many situations. For example, when one speaks of black holes in centers of galaxies, one does not refer to event horizons. The idealization seems unsuitable also for black hole mechanics and statistical mechanical calculations of entropy. First, in ordinary equilibrium statistical mechanics, one assumes only that the system under consideration is stationary, not the whole universe. Second, thermodynamic considerations are known to apply also to cosmological horizons [3]. Thus, it seems desirable to replace event horizons by a quasilocal notion and develop a detailed framework tailored to diverse applications, from numerical relativity to quantum gravity, without the assumption of global stationarity. The purpose of this Letter is to present such a framework.

Specifically, we will provide a set of quasilocal boundary conditions which define an isolated horizon Δ representing, for example, the last stages of a collapse or a merger, and focus on space-time regions admitting such horizons as an inner boundary. Although the boundary conditions are motivated purely by geometric considerations, they lead to a well-defined action principle and Hamiltonian framework. This, in turn, leads to a definition of the horizon mass M_Δ and angular momentum J_Δ. These quantities refer only to structures intrinsically available on Δ, without any reference to infinity, and yet lead to a generalization of the familiar laws of black hole mechanics. We will also introduce invariantly defined coordinates near Δ and a Bondi-type expansion of the metric. Finally, our present boundary conditions allow distorted and rotating horizons and are thus significantly weaker than those introduced in earlier papers [4]. With this extension, the framework becomes a robust new tool in the study of classical and quantum black holes.

For brevity, in the main discussion we will restrict ourselves to the Einstein-Maxwell theory in four space-time dimensions. Throughout, $\triangleq$ will stand for equality restricted to Δ; an arrow under an index will denote pullback of that index to Δ; V^a will be a generic vector field tangential to Δ, and $\tilde{V}^a$ any of its extensions to space-time. The electromagnetic potential and fields will be denoted by boldfaced letters. All fields are assumed to be smooth, and bundles, trivial. For details, generalizations, and subtleties, see [4–7].

Definition: A submanifold Δ of a space-time (M, g_{ab}) is said to be an *isolated horizon* if (i) *it is topologically* $S^2 \times R$*, null, with zero shear and expansion.* This condition implies, in particular, that the space-time ∇ induces a unique derivative operator D on Δ via $D_a V^b := \nabla_{\underleftarrow{a}} \tilde{V}^b$. (ii) $(\mathcal{L}_l D_a - D_a \mathcal{L}_l) V^b \triangleq 0$ and $\mathcal{L}_l \mathbf{A}_{\underleftarrow{a}} \triangleq 0$ *for* some *null normal* l *to* Δ; and, (iii) *field equations hold at* Δ.

All these conditions are *local* to Δ. The first two imply that the intrinsic metric and connection on Δ are "time independent" and specify the precise sense in which Δ is "isolated." Every Killing horizon which is topologically $S^2 \times R$ is an isolated horizon. However, in general, space-times with isolated horizons need not admit *any* Killing field even in a neighborhood of Δ. The local existence of such space-times was shown in [8]. A global example is provided by Robinson-Trautman space-times which admit an isolated horizon but have radiation in *every* neighborhood of it [9]. Finally, on a general Δ, the null normal l of (ii) plays a role analogous to that of the Killing field on a Killing horizon. Generically, l satisfying (ii) is unique up to a *constant* rescaling $l \to cl$. (In particular, this is true of the Kerr family.) We will denote by $[l]$ the equivalence class of null normals satisfying (ii). One cannot hope to eliminate this constant rescaling freedom because it exists already on (local) Killing horizons.

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Geometry of isolated horizons.— Although the boundary conditions are rather weak, they have surprisingly rich consequences. We now summarize the most important ones.

(1) Intrinsic geometry: l is a symmetry of the degenerate, intrinsic metric $q_{ab} := g_{\underline{ab}}$ of Δ; $\mathcal{L}_l q_{ab} \triangleq 0$. Δ is naturally equipped with a 2-form ϵ_{ab}, the pullback to Δ of the volume 2-form on the 2-sphere of integral curves of l, satisfying $\epsilon_{ab} l^b \triangleq 0$ and $\mathcal{L}_l \epsilon_{ab} \triangleq 0$. The area of any cross section S is given by $\oint_S \epsilon$ and is the same for all cross sections. We will denote it by a_Δ.

(2) Connection coefficients: l is geodesic and free of divergence, shear, and twist. Hence there exists a 1-form ω on Δ such that $\nabla_{\underline{a}} l^b = \omega_a l^b$. The surface gravity $\kappa_{(l)}$ defined by l is given by $\kappa_{(l)} = \omega_a l^a$. The boundary conditions imply $\kappa_{(l)}$ is constant on Δ [6]. Thus, the zeroth law holds. Similarly, the electromagnetic potential $\Phi_{(l)} = -\mathbf{A}_a l^a$ is constant on Δ [6]. Note, however, that other connection components or the scalar curvature of the intrinsic metric q_{ab} need not be constant; the horizon may be distorted arbitrarily.

(3) Weyl curvature: Let us pick an l in $[l]$ and construct a null tetrad $l, n, m, \overline{m}$ on Δ. Here $m, \overline{m}$ are chosen to be tangential to Δ and thus n is transverse. Then, the Weyl components $\Psi_0 = C_{abcd} l^a m^b l^c m^d$ and $\Psi_1 = C_{abcd} l^a m^b l^c n^d$ vanish, implying that there is no flux of gravitational radiation across Δ and the Weyl tensor at Δ is of Petrov type II [6]. Hence $\Psi_2 := C_{abcd} l^a m^b \overline{m}^c n^d$ is gauge invariant. Its imaginary part is determined by ω via $d\omega = 2\,\mathrm{Im}\Psi_2 \epsilon$, and encodes the gravitational contribution to the horizon angular momentum is determined entirely by $\mathrm{Im}\Psi_2$. While Ψ_2 is time independent on the horizon, in general, $\Psi_3 = C_{abcd} l^a m^b \overline{m}^c n^d$ and $\Psi_4 = C_{abcd} n^a \overline{m}^b n^c \overline{m}^d$ are not [7].

(4) A natural foliation: Let us consider the nonextremal case when $\kappa_{(l)}$ is nonzero. Then, Δ admits a natural foliation, thereby providing a natural "horizon rest frame" [7]. The 2-sphere cross sections of the horizon defined by this foliation are analogous to the "good cuts" that null infinity admits in the absence of Bondi news. Therefore, we will refer to them as *good cuts* of the horizon. If there is no gravitational angular momentum, i.e., if $\mathrm{Im}\Psi_2 \triangleq 0$, then $d\omega$ vanishes. Hence, there exists a function ψ on Δ with $\omega \triangleq d\psi$. Since $\mathcal{L}_l \psi \triangleq \omega \cdot l \triangleq \kappa$ is constant on Δ, the $\psi \triangleq$ constant surfaces foliate Δ. In the general case, the argument is more involved but the foliation is again determined invariantly by the geometrical structure of Δ.

(5) Symmetries of Δ: In view of our main definition, the symmetry group $\mathcal{G}_\Delta$ of a given isolated horizon is the subgroup of the diffeomorphism group of Δ which preserves $[l]$, q_{ab}, D, and $\mathbf{A}_{\underline{a}}$. Since q_{ab}, D, and $\mathbf{A}_{\underline{a}}$ can vary from one isolated horizon to another, $\mathcal{G}_\Delta$ is not canonical. For simplicity, let us again restrict ourselves to the nonextremal case $\kappa_{(l)} \neq 0$. Then, isolated horizons fall into three *universality classes* [7]: (I) $\dim\mathcal{G}_\Delta = 4$: In this case, q_{ab} is spherically symmetric, good cuts are invariant under the natural SO(3) action, and $\mathcal{G}_\Delta$ is the direct product of SO(3) with translations along l; (II) $\dim\mathcal{G}_\Delta = 2$: In this case, q_{ab} is axisymmetric, the general infinitesimal symmetry ξ^a has the form $\xi^a \triangleq c l^a + \Omega \varphi^a$, where c, Ω are arbitrary constants on Δ and φ is a rotational vector field tangential to good cuts; and, (III) $\dim\mathcal{G}_\Delta = 1$: In this case, the infinitesimal horizon symmetry has the form $\xi^a = c l^a$. In case I, the horizon is undistorted and nonrotating while case III allows general distortion and rotation.

Extracting physics.— The isolated horizon framework can be used to extract invariant physical information in the strong field region near black holes, formed by gravitational collapse or merger of compact objects. At a sufficiently late time, the space-time would contain an (approximate) isolated horizon Δ. In the most interesting case, Δ would be of universality class II above. We will now focus on this class and comment on other cases at the end of this Letter. First, we can ask for the angular momentum and mass of Δ. Recall that, for asymptotically flat space-times without internal boundaries, one obtains expressions of the ADM mass M_∞ and angular momentum J_∞ using a Hamiltonian framework. This strategy can be extended to the present case (see below). When constraints are satisfied, the total Hamiltonian is now a sum of two surface terms, one at infinity and the other at Δ. The terms at infinity again yield M_∞ and J_∞. General arguments lead one to interpret the surface terms at Δ as the horizon mass M_Δ and angular momentum J_Δ. We have [7]

$$J_\Delta = -\frac{1}{4\pi G} \oint_S f(\mathrm{Im}\Psi_2 \epsilon + 2G\,\mathrm{Im}\phi_1 {}^\star\mathbf{F}), \qquad (1)$$

where S is any 2-sphere cross section of Δ, f is related to φ by $D_a f = \epsilon_{ba}\varphi^b$, and $\mathrm{Im}\phi_1 = -(i/2)\mathbf{F}_{ab} m^a \overline{m}^b$ is a Newman-Penrose component of the Maxwell field. In a vacuum, axisymmetric space-time, $J_\Delta = J_\infty$. However, in general, the two differ by the angular momentum in the gravitational radiation and the Maxwell field in the region between Δ and infinity. Even in the presence of such radiation, the horizon mass is given by [7]

$$M_\Delta = \frac{1}{2GR_\Delta}[(R_\Delta^2 + GQ^2)^2 + 4G^2 J_\Delta^2]^{1/2}, \qquad (2)$$

where R_Δ is the horizon radius, given by $a_\Delta = 4\pi R_\Delta^2$, and $Q_\Delta = -\frac{1}{4\pi}\oint_S {}^\star\mathbf{F}$ is the horizon charge. Somewhat surprisingly, M_Δ has the same dependence on area, angular momentum, and charge as in the Kerr-Newman family [provided J_Δ is defined via (1)]. However, this is a *result* of the calculation, not an assumption. While in any Kerr-Newman space-time $M_\infty = M_\Delta$, in general M_Δ is different from M_∞. Under certain physically reasonable assumptions on the behavior of fields near future timelike infinity i^+, one can show that the difference is the energy radiated across $\mathcal{I}^+$ by gravitational and electromagnetic waves.

If $\kappa_{(l)} \neq 0$, using good cuts one can introduce (essentially) invariant coordinates and tetrads in a neighborhood of Δ, irrespective of the universality class. Fix an l in $[l]$. Let v, θ, ϕ be coordinates on Δ such that $\mathcal{L}_l v \triangleq 1$ and good cuts are given by $v \triangleq$ const. Let n^a be the unique future-directed null vector field which is orthogonal to the good cuts and normalized so that $l \cdot n \triangleq -1$. Consider past null geodesics emanating from the good cuts, with $-n^a$ as their tangent. Finally, define r via $\mathcal{L}_n r = -1$ and $r = r^o$ on Δ, and Lie drag v, θ, ϕ along n^a. We now have a natural set of coordinates (r, v, θ, ϕ); the only freedom is in the *initial* choice of (θ, ϕ) and in adding constants to r, v. Next, let us parallel transport $l, m, \overline{m}$ along n to obtain a null tetrad in this neighborhood. The tetrad is unique up to local m-$\overline{m}$ rotations *at* Δ. Now, assuming the vacuum equations hold in this neighborhood, one can give a Bondi-type expansion for the metric components in powers of $(r$-$r^o)$ to any desired order. For example, retaining terms to first order, we have [7]

$$\begin{aligned} g_{ab} = {} & 2m^o_{(a}\overline{m}^o_{b)} + 2r_{,(a}v_{,b)} - (r - r^0) \\ & \times [4\mu^o m^o_{(a}\overline{m}^o_{b)} + 2\lambda^o m^o_{(a}m^o_{b)} + 2\overline{\lambda}^o \overline{m}^o_{(a}\overline{m}^0_{b)} \\ & + 2v_{,(a}(2\omega_{b)} - \kappa_{(l)}v_{,b)})] + O(r - r^o)^2, \end{aligned}$$

where quantities with the superscript o are evaluated on Δ, and the Newman-Penrose spin coefficients are defined as $\mu = m^a\overline{m}^b\nabla_a n_b$ and $\lambda = \overline{m}^a\overline{m}^b\nabla_a n_b$. Using the boundary conditions and field equations, at the horizon these spin coefficients can be expressed in terms of the dyad $m^o, \overline{m}^o$ and the 1-form ω_a on any one good cut [7]. The coefficient of $(r - r^o)^n$ in the expansion is expressible in terms of these fields *and* the $(n$-2)th radial derivative of Ψ_4, evaluated on Δ.

The null surfaces $v =$ const are invariantly defined. Therefore (modulo the small freedom mentioned above) the tetrad components of the Weyl tensor on these surfaces are gauge invariant. This property will be useful in physically interpreting the outcomes of numerical simulations of mergers of compact objects. For example, it will enable a gauge invariant comparison between the radiation fields $|\Psi_4|$ created in two simulations, say with somewhat different initial conditions. Finally, one can give a systematic procedure to extend any infinitesimal symmetry $t^a \triangleq cl^a + \Omega\varphi^a$ of Δ to a "potential Killing field" $\tilde{t}^a$ in a neighborhood [7]. If the space-time does admit a Killing field ξ^a which coincides with t^a on Δ, then ξ^a must equal $\tilde{t}^a$ in the neighborhood. Again, since they are defined invariantly, the vector fields $\tilde{t}^a$ can be useful to extract physics from the strong field geometry.

Finally, note that all this structure—particularly the definitions of M_Δ and J_Δ—is defined intrinsically, using local geometry of the *physical* space-time under consideration. To extract physical information, one does not have to embed this space-time in a Kerr solution which presumably approximates the physical, near horizon geometry at late times. In practice this is a significant advantage because the embedding problem can be very difficult: typically, one knows little about the desired form of the metric or the values of the Kerr parameters to use in the embedding. Furthermore, one does not have a quantitative control on precisely how the physical near-horizon geometry is to approach Kerr.

Isolated Horizon Mechanics.—We already saw that the zeroth law holds on all isolated horizons. Let us consider the first law: $\delta M = (\kappa/8\pi G)\delta a + \Omega\delta J + \Phi\delta Q$. In the standard, stationary context the law is somewhat "hybrid" in that M and J are defined at infinity, a at the horizon, and κ, Ω, and Φ are evaluated at the horizon but refer to the normalization of the Killing field carried out at infinity. In the nonstationary context now under consideration, there are two additional problems: due to the presence of radiation, M_∞ and J_∞ have little to do with the horizon mass and since we no longer have a global Killing field, there is an ambiguity in the normalization of κ and Ω.

As in [10], our strategy is to arrive at the first law through a Hamiltonian framework, but now adapted to the isolated horizon boundary conditions. For brevity, we will again focus on the physically most interesting universality class II. Let us fix on the (abstract) isolated horizon boundary Δ a rotational vector field φ^a. Consider the space Γ of asymptotically flat solutions to the Einstein-Maxwell equations for which Δ is an isolated horizon innerboundary with symmetry φ^a. Γ will be our covariant phase space [6,7]. Denote by $\tilde{\varphi}^a$ any extension of φ^a which is an asymptotic rotational Killing field at spatial infinity. Then, one can show that the vector field $\delta_{\tilde{\varphi}}$ on Γ defined by the Lie derivative of basic fields along $\tilde{\varphi}^a$ is a phase space symmetry, i.e., Lie drags the symplectic structure. Its generator is given by [7]

$$H_{\tilde{\varphi}} = J_\infty - J_\Delta\,,$$

where J_Δ is given by (1). Hence, it is natural to interpret (1) as the horizon angular momentum.

To define the horizon energy, one needs to select a "time translation." On Δ, it should coincide with a horizon symmetry $t^a \triangleq cl^a + \Omega\varphi^a$. While c, Ω are constants on Δ, in the phase space we must allow them to vary from one solution to another. (In the numerical relativity language, we must allow t^a—or, the lapse and shift at Δ—to be *live.*) For, unlike at infinity, the 4-geometries under consideration do not approach a fixed 4-geometry at Δ, whence it is not *a priori* obvious how to pick the *same* time translation for all geometries in the phase space. Let $\tilde{t}^a$ be any extension of t^a to the whole space-time which approaches a *fixed* time translation at infinity. We can ask if the corresponding $\delta_{\tilde{t}}$ is a phase space symmetry. The answer is rather surprising: yes, *if and only if there exists a function E^t_Δ on the phase space, involving only the horizon fields, such that the first law,*

$$\delta E^t_\Delta = \frac{\kappa_{(t)}}{8\pi G}\,\delta a_\Delta + \Omega_t\delta J_\Delta + \Phi_{(t)}\delta Q\,, \tag{3}$$

holds [6,7]. Thus, not only does the isolated horizon framework enable one to extend the first law beyond the stationary context, but it also brings out its deeper role: the first law is a necessary and sufficient condition for a consistent Hamiltonian evolution.

However, there are many choices of t^a on the horizon for which this condition can be met, each with a corresponding time evolution, horizon energy function, and first law. Can we make a canonical choice of t^a? In the Einstein-Maxwell theory, the answer is in the affirmative. The requirement that the (live) vector field $\tilde{t}^a$ coincide, on each Kerr-Newman solution, with that stationary Killing field which is unit at infinity *uniquely* fixes t^a on the isolated horizon of every space-time in the phase space. With this canonical choice, say $t = t_o$, in Einstein-Maxwell theory we can define the horizon mass to be

$$M_\Delta = E_\Delta^{t_o}.$$

Then, M_Δ is given by (2).

We will conclude with three remarks.

(1) We focused our discussion on the physically most interesting universality class II. Class I was treated in detail in [4,5] and is a special case of nonrotating, class III horizons discussed in [6]. All these cases have been analyzed in detail. However, the current understanding of class III with rotation ($\mathrm{Im}\Psi_2 \neq 0$) is rather sketchy.

(2) The framework that led us to the zeroth and first laws can be easily extended to other space-time dimensions. The $2 + 1$-dimensional case has already been analyzed in detail [11] and has some special interesting features in the context of a negative cosmological constant. In the nonrotating, class III case, dilaton and Yang-Mills fields have also been incorporated [4–6]. In the Yang-Mills case, although the zeroth and first laws can be proved, the analog of the mass formula (2) is not known because one does not have as much control on the space of all stationary solutions. Nonetheless, the framework has been used to derive new relations between masses of static black holes with hair and their solitonic analogs in Einstein-Yang Mills theory [5,6]. More importantly, as is well known, the standard no-hair theorems fail in this case and the framework has been used to conjecture new no-hair theorems tailored to isolated horizons rather than infinity [5].

(3) In the nonrotating case ($\mathrm{Im}\Psi_2 \triangleq 0$), the framework has been used to carry out a systematic and detailed entropy calculation using nonperturbative quantum gravity [12]. The analysis encompasses all black holes without any restriction of near extremality made in string theory calculations. Furthermore, it also naturally incorporates the cosmological horizons to which thermodynamic considerations are known to apply [3]. Recently, subleading corrections to entropy have also been calculated [13]. However, the nonperturbative quantization scheme faces a quantization ambiguity—analogous to the θ ambiguity in QCD—which permeates all these calculations. Its role is not fully understood. Carlip [14] and others have suggested the use of horizon symmetries in entropy calculations and this approach could shed light on the quantization ambiguity and relate the analysis of [12] to conformal field theories. Conversely, the isolated horizon framework may offer a more systematic avenue for implementing Carlip's ideas. Finally, since rotation has now been incorporated in the classical theory [7], one can hope to extend the entropy calculation to this case.

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