Robustness and Diffusion of Pointer States

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(Received 18 May 2000)

Classical properties of an open quantum system emerge through its interaction with other degrees of freedom (decoherence). We treat the case where this interaction produces a Markovian master equation for the system. We derive the corresponding distinguished local basis (pointer basis) by three methods. The first demands that the pointer states mimic as closely as possible the local nonunitary evolution. The second demands that the local entropy production be minimal. The third imposes robustness on the inherent quantum and emerging classical uncertainties. All three methods lead to localized Gaussian pointer states, their formation and diffusion being governed by well-defined quantum Langevin equations.

PACS numbers: 03.65.Bz, 05.60.Gg

The program of decoherence has been very successful in explaining the classical appearance of macroscopic or mesoscopic quantum systems, both theoretically and experimentally [1]. The interaction of a quantum system with its environment leads in these cases to a delocalization of phase relations in the full configuration space of system plus environment, preventing them from being observed locally, i.e., at the system itself. Thereby the environment distinguishes a certain preferred basis for the system, which can be used to describe the apparent classical behavior (*pointer basis* [2]). An important question is how the pointer basis can be determined and whether it is *uniquely* fixed. It would then be possible to give a unique decomposition of the reduced density matrix into an apparent ensemble of wave functions.

No unique rules have so far been adopted to calculate the pointer states in the general case. In [3] the suggestion was made that the pointer basis (there called collection of "memory states") is characterized by its *robustness* (there called "dynamical stability"). A first quantitative measure to investigate the dynamical stability is the rate of deseparation introduced in [4]—it measures how fast a quantum system becomes entangled with environmental degrees of freedom. In a model consisting of harmonic oscillators, it was shown that coherent states are the stablest states and therefore can be considered as pointer states [1,4]. A different measure for robustness is the "predictability sieve" put forward in [5]. The pointer basis is there distinguished by the property of having the least production rate for local entropy during the coupling to the environment. In the case of harmonic oscillators, this again leads to the coherent states as the pointer basis [6]. At least for such simple systems, the rate of deseparation and predictability sieve are roughly equivalent measures [1].

On the other hand, the theory and formalism of quantum state diffusion (QSD) were put forward to attribute random wave functions for local systems, which satisfy an appropriate Langevin equation [7]. These wave functions are known to be related to possible continuous measurements [8] as well as to decoherent histories [9] of the given local system. But even if we take them as wave functions of mere formal meaning (since subsystems do not, in general, possess their own pure states), the question arises whether there is any connection between these states and the pointer states, in cases where the local system exhibits classical properties. We shall show that there is, in fact, such a connection—pointer basis and QSD basis are substantially the same. For this aim, we shall also present below a new, alternative, derivation of QSD.

In the following we shall consider the dynamics of the reduced density matrix, $\hat{\rho}(t)$, of a system interacting with a certain decohering environment. Ideal *pointer states* (described by a fixed set of projectors $\hat{P}_n$) are characterized by the fact that $\hat{\rho}(t)$ can be decomposed as

$$\hat{\rho}(t) \to \sum_n f_n \hat{P}_n, \qquad t \gg t_D\,, \tag{1}$$

for a generic initial state $\hat{\rho}(0)$, where t_D is the decoherence time. The weights $\{f_n\}$ correspond to a normalized probability distribution. The pointer states $\{\hat{P}_n; n = 1, 2, \ldots\}$ form in this case an orthogonal system. For macroscopic systems, t_D is extremely short [1,10]. More generally, one would expect

$$\hat{\rho}(t) \to \int f(\Gamma)\hat{P}(\Gamma)\, d\Gamma, \qquad t \gg t_D\,, \tag{2}$$

where $f(\Gamma)$ is a probability distribution over the pointer states $\hat{P}(\Gamma)$ which project now on an overcomplete set of pure states (the above-mentioned coherent states provide an example for this). The pointer states in (1) or (2) result after an explicit interaction with the environment is taken into account [1].

 0031-9007/00/85(17)/3552(4)$15.00

In many cases, the effects of decoherence can be described by the following Markovian master equation [1],

$$\frac{d\hat{\rho}(t)}{dt} \equiv \mathcal{L}\,\hat{\rho}(t) = -\frac{i}{2m}[\hat{p}^2, \hat{\rho}(t)] - \frac{D}{2}[\hat{x}, [\hat{x}, \hat{\rho}(t)]], \tag{3}$$

where D describes the strength of the interaction with the environment. Such an equation arises, for example, in situations where environmental degrees of freedom scatter off a macroscopic object and localize it by carrying away quantum correlations with the object [1,10]. Applying the concept of predictability sieve would mean to minimize the local production of "linear entropy" $S(t) = 1 - \mathrm{tr}\hat{\rho}^2(t)$. This does not give a unique answer, since the result depends explicitly on t. In the oscillator case, one has therefore calculated the time-integrated entropy production [1,6]. If one considers the initial entropy production rate $\dot{S}(0)$, assuming the initial state $\hat{\rho}(0)$ of the subsystem to be a pure state $\hat{P}(\Gamma)$, one finds from (3) that $\dot{S}(0) = D(\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2) \equiv D\sigma^2$, where the expectation value refers to the initial state. This rate would be minimized if the pointer wave functions were delta functions. However, their spread σ increases dynamically due to the kinetic term in (3). Therefore, very narrow wave functions do not produce minimum entropy on a finite time scale and thus cannot correspond to pointer states. The (coherent) unitary spreading and the (incoherent) nonunitary localizing terms of the master equation (3) are competing with each other. For a wave function of characteristic width σ, the above two effects are approximately balanced for the "equilibrium width" [1,10,11]

$$\sigma_0 \sim (Dm)^{-1/4}. \tag{4}$$

It is then reasonable to conjecture that, in the spirit of predictability sieve and of dynamical robustness, σ_0 will be the characteristic width of the pointer states. This is, in fact, what we shall show by using three different methods, all invoking a principle of robustness.

The first method goes as follows. Let us allow for the pointer state $\hat{P}(\Gamma)$ a certain natural time dependence such that it may initially evolve as closely as possible along the true state $\hat{\rho}$ satisfying the master equation (3), and then reach a stationary state. We introduce the "speed" v describing the departure of the states $\hat{P}(\Gamma)$ from $\hat{\rho}$ in the Hilbert-Schmidt norm,

$$v^2 = \mathrm{tr}\left[\frac{d}{dt}\hat{P}(\Gamma) - \mathcal{L}\,\hat{P}(\Gamma)\right]^2. \tag{5}$$

The smaller v, the greater is the robustness of the pointer states $\hat{P}(\Gamma)$. Hence one defines the optimum drift of the *pure* pointer state $\hat{P}(\Gamma) \equiv \psi\psi^\dagger$ by minimizing v [12]. This is given by the nonlinear equation [12,13]

$$\dot{\psi} = (\mathcal{L}\,\psi\psi^\dagger)\psi - \langle\mathcal{L}\,\psi\psi^\dagger\rangle\psi\,. \tag{6}$$

This result is valid for all kinds of Markovian subdynamics. In our special case (3), it yields the following nonlinear wave equation:

$$\dot{\psi} = -\frac{i}{2m}\hat{p}^2\psi - \frac{D}{2}[(\hat{x} - \langle\hat{x}\rangle)^2 - \sigma^2]\psi\,. \tag{7}$$

As shown in [14], this equation has a stationary solution which is *unique* up to Galilean transformations. The wave function of the fiducial stationary state is the complex Gaussian wave packet

$$\psi_0(x) = (\alpha_R/2\pi)^{1/4}\exp(-\alpha x^2/4)\,, \tag{8}$$

with parameter

$$\alpha \equiv \alpha_R + i\alpha_I = (1 - i)\sqrt{2Dm}\,. \tag{9}$$

The principle of "Hilbert-Schmidt robustness" has thus singled out unique Gaussian pointer states as the robust pure states closest to the true nonunitary local dynamics. The exact width $\sigma \equiv 1/\sqrt{\alpha_R}$ confirms the heuristic estimate (4). Accordingly, we restrict our further discussion to pointer states $\hat{P}(\Gamma)$ with Gaussian wave functions and make the ansatz

$$\psi_\Gamma(x) = (\alpha_R/2\pi)^{1/4}\exp[-\alpha(x - \overline{x})^2/4 + i\overline{p}(x - \overline{x})], \tag{10}$$

where $\Gamma \equiv (\overline{x}, \overline{p})^T$ has been understood. For later purposes, we calculate the correlation matrix $\mathbf{C}$,

$$\mathbf{C} \equiv \langle\psi_0|\begin{pmatrix} \hat{x}^2 & (\hat{x}\hat{p} + \mathrm{H.c.})/2 \\ (\hat{x}\hat{p} + \mathrm{H.c.})/2 & \hat{p}^2 \end{pmatrix}|\psi_0\rangle = \frac{1}{\alpha_R}\begin{pmatrix} 1 & -\alpha_I/2 \\ -\alpha_I/2 & |\alpha|^2/4 \end{pmatrix}, \tag{11}$$

of the quantum uncertainties of the canonical observables in the pointer state itself, where ψ_0 denotes the fiducial state (8). It was shown in [10] that the states diagonalizing $\hat{\rho}$ exactly are the harmonic oscillator eigenfunctions which are very broad, while narrow eigenfunctions are apparently obtained only for discrete systems. In contrast to these, the above pointer states are well localized. For example, in the situation of a small dust particle ($m = 10^{-14}$ g) scattered by air molecules one has $D \sim 10^{32}\ \mathrm{cm^{-2}\,s^{-1}}$ [10] and therefore $\sigma_0 \approx (Dm)^{-1/4} \approx 10^{-11}$ cm and $t_D \approx \sqrt{m/D} \approx 10^{-10}$ s.

We now come to the second method. As a preparation, we shall discuss the reduced dynamics of the local system in the basis given by (10). We allow temporarily the parameter α to take an arbitrary complex value, and then derive again a distinguished value. If one allows a "natural" time dependence for the probability distribution $f(\Gamma; t)$ of the pointer, the asymptotic condition (2) can be turned into an exact identity:

$$\hat{\rho}(t) = \int f(\Gamma; t)\hat{P}(\Gamma)\,d\Gamma, \qquad t > t_D\,, \tag{12}$$

where $d\Gamma \equiv d\overline{x}d\overline{p}/2\pi$. This important fact will be proven elsewhere [15]. It generalizes the corresponding statement made in [14] for the specific value (9) of α as well as the asymptotic statement proved in [16,17].

From (3) and (12) one can derive an evolution equation for $f(\Gamma;t)$,

$$\frac{df(\Gamma;t)}{dt} = -\frac{\overline{p}}{m}\,\partial_{\overline{x}} f(\Gamma;t) + \frac{1}{2}\,[D_{pp}\partial^2_{\overline{p}\overline{p}} + D_{xx}\partial^2_{\overline{x}\overline{x}} + 2D_{px}\partial^2_{\overline{p}\overline{x}}] f(\Gamma;t)\,, \tag{13}$$

where the elements of the diffusion matrix are given by

$$\mathbf{D} \equiv \begin{pmatrix} D_{xx} & D_{xp} \\ D_{px} & D_{pp} \end{pmatrix} = \begin{pmatrix} -\alpha_I/m\alpha_R & |\alpha|^2/4m\alpha_R \\ |\alpha|^2/4m\alpha_R & D \end{pmatrix}. \tag{14}$$

To find a formal solution of (13), we use the Fourier representation $\tilde{f}(\tilde{\Gamma};t) = \int f(\Gamma;t)\exp[i(\tilde{x}\overline{p} - \tilde{p}\overline{x})]\,d\Gamma$ with $\tilde{\Gamma} = (\tilde{x},\tilde{p})^T$. Equation (13) then leads to

$$\frac{d\tilde{f}(\tilde{\Gamma};t)}{dt} = -\frac{\tilde{p}}{m}\,\partial_{\tilde{x}}\tilde{f}(\tilde{\Gamma};t) - \frac{1}{2}\,|\mathbf{D}|\,[\tilde{\Gamma}^T\mathbf{D}^{-1}\tilde{\Gamma}]\tilde{f}(\tilde{\Gamma};t)\,, \tag{15}$$

where $|\mathbf{D}|$ denotes the determinant of $\mathbf{D}$. The solution takes the form

$$\tilde{f}(\tilde{\Gamma};t) = \exp\!\left[-\frac{t}{2}\,\tilde{\Gamma}^T\mathbf{G}(t)\tilde{\Gamma}\right]\tilde{f}(\tilde{x} - \tilde{p}t/m, \tilde{p}; 0)\,. \tag{16}$$

By substitution into (15) one obtains explicitly the matrix of time-dependent coefficients,

$$\mathbf{G}(t) = \begin{pmatrix} D_{pp} & -D_{xp} + D_{pp}t/2m \\ -D_{xp} + D_{pp}t/2m & D_{xx} - D_{xp}t/m + D_{pp}t^2/3m^2 \end{pmatrix}. \tag{17}$$

Equation (13) can be interpreted as a Fokker-Planck equation provided the diffusion matrix $\mathbf{D}$ is non-negative. Then the weight function $f(\overline{x},\overline{p};t)$ of the pointer states $\hat{P}(\overline{x},\overline{p})$ will drift according to the free-particle dynamics. At the same time *the state of the system will diffuse over the pointer states* $\hat{P}(\overline{x},\overline{p})$. We can now implement the predictability sieve and minimize the production rate for linear entropy by minimizing the width of the Gaussian pointer states. In other words, we maximize α_R under the condition that the diffusion matrix be non-negative. The condition that $\mathbf{D}$ has a non-negative determinant leads to the condition $\alpha_R^4 + 2\alpha_R^2\alpha_I^2 + 16Dm\alpha_R\alpha_I + \alpha_I^4 \le 0$ which can be fulfilled only if $\alpha_I < 0$, since $\alpha_R > 0$ for (10) to be normalizable. Introducing dimensionless polar coordinates R,ϕ by $\alpha \equiv \sqrt{Dm}\,R\exp(i\phi)$, this condition reads $R^2 + 8\sin 2\phi \le 0$. The maximum for $\alpha_R = \sqrt{Dm}\,R\cos\phi$ is reached if the equality sign holds, since one could otherwise increase R by holding ϕ fixed and thus increase α_R. Maximizing α_R under the condition $R^2 = -8\sin 2\phi$ then yields for α the following value α_s distinguished by the predictability sieve:

$$\alpha_s = 3^{1/4}(\sqrt{3} - i)\sqrt{Dm}\,, \tag{18}$$

which coincides, up to a small deviation in the numerical coefficients, with the value (9) following from the criterion of Hilbert-Schmidt robustness. This above slight departure of α_s might be related to the fact that the given form of predictability sieve predicts a degenerate diffusion matrix $\mathbf{D}$. We think, however, that the emerging incoherent uncertainties due to the pointer state diffusion must be made proportional to the quantum uncertainties already present in the pointer states itself. The "robustness of uncertainties" demands that the matrix $\mathbf{C}$ (11) of quantum correlations *be proportional* to the diffusion matrix $\mathbf{D}$ (14) of the corresponding classical coordinates for the pointer. From the condition that $\mathbf{C} = \text{const} \times \mathbf{D}$ we then obtain again the standard value (9) for α, while $\mathbf{C} = m/2D \times \mathbf{D}$.

We shall now discuss our last method to determine the pointer basis, which will involve quantum state diffusion. As we see from (12) and (13), the quantum state of the system, when expanded as a mixture of pointer states, performs diffusion after the decoherence time has elapsed. This diffusion will, by construction, preserve the shape (8) of the Gaussian wave packet, and only its center will walk randomly. It is then natural to ask whether there is a generic QSD process which, first, applies to *generic* pure initial states and, second, tends to the above specific diffusion process for $t \gg t_D$.

As is well known, the Fokker-Planck equation (13) is equivalent to the Itô-Langevin equation [18]

$$d\Gamma = V dt + dX\,, \tag{19}$$

where $V = (\overline{p}/m, 0)$, and $dX = (d\xi, d\pi)$ is the increment of a zero-mean Gaussian white noise with correlation matrix $\mathbf{D}\,dt$. In case of phase-space diffusion the use of the Itô-Langevin formalism instead of the Fokker-Planck formalism is a matter of taste. But the diffusion of the corresponding pointer states ψ_Γ would be quite awkward in the Fokker-Planck formalism. We thus choose the Itô formalism and apply (19) to the Gaussian pointer states (10). This leads to, substituting $\langle\hat{x}\rangle = \overline{x}$ and $\langle\hat{p}\rangle = \overline{p}$,

$$d\psi = -\frac{i}{2m}\,\hat{p}^2\psi\,dt - \frac{D}{2}\,(\hat{x} - \langle\hat{x}\rangle)^2\psi\,dt + (\hat{x} - \langle\hat{x}\rangle)\psi\,dz\,, \tag{20}$$

where the index Γ has been skipped. The deterministic part of the evolution is governed, up to normalization, by the same nonlinear wave equation (7) which we had obtained from the Hilbert-Schmidt robustness, while the random part is driven by the complex Gaussian white noise

$$dz = \frac{\alpha}{2}\,d\xi + i\,d\pi\,. \tag{21}$$

Since the correlation matrix of $dX = (d\xi, d\pi)$ is $\mathbf{D}\,dt$, (14) and (21) yield

$$M[dz\,dz^\star] = D\,dt \tag{22}$$

for the mean of the Hermitian correlation, independent of α. But the correlation $M[dz\,dz]$ still depends on α. A

most remarkable feature of (20) is that any reference to the phase-space variables $\Gamma = (\overline{x}, \overline{p})$ has been canceled. For this reason we have omitted the subscript Γ from ψ and extend the validity of the equation to *arbitrary* initial state vectors. It is possible to prove [16,19–21] that, starting from whatever initial state $\psi(0)$, the random solution $\psi(t)$ will tend to be the Gaussian pointer state $\psi_{\Gamma(t)}$, where $\Gamma(t)$ is governed by the diffusion process (19). Equation (20) is called the Itô-Schrödinger equation of QSD.

Since the free parameter α still appears in $M[dz\, dz]$, we are left with the nonuniqueness problem of the QSD equations. If one, however, chooses the distinguished value (9) of α, one finds, using (21), the simple result

$$M[dz\, dz] = M[dz^\star\, dz^\star] = 0\,, \tag{23}$$

distinguishing a unique QSD. Historically, this unique QSD was in the Fokker-Planck formalism singled out by certain invariance considerations [22,23]. The Itô-Schrödinger equation (20) with the complex Gaussian white noise [(22) and (23)] has become the dominating formalism of *standard* QSD theory [7] extended for arbitrary Markovian reduced dynamics. Applying exact forms of robustness criteria we have thus obtained a unique QSD which leads to stationary Gaussian pointer states for $t \gg t_D$, whose centers undergo a diffusion process. With heuristic forms of robustness one could have chosen other QSD equations like in [19] ($\alpha_R = 2\sqrt{Dm}$) or [16] ($\alpha_R = \sqrt{Dm}/2\sqrt{2}$). The recent proposal of "maximal survival probability" from [24] differs from our first method and does not lead to (23) [15].

In conclusion, we have demonstrated that three different methods of dynamical robustness lead to an essentially *unique* local pointer basis in case of Markovian local dynamics. The corresponding pointer states follow the classical trajectories up to a tiny random diffusion. Well-defined stochastic differential equations, known from the theory of quantum state diffusion, govern both the formation and the diffusion of pointer states. These states can thus be used to characterize local quasiclassical properties. The pointer states are not an absolute property of the system in itself, but characterize only certain stability properties with respect to interactions with the environment: They are least sensitive to quantum entanglement, which is why interference terms between them cannot be noticed by local observers. They possess thus meaning with respect to an observer-related branch of the total wave function or a component corresponding to a potential fundamental collapse [1,25], while the interaction with the environment is encoded in the choice of our master equation (3).

We thank the Institute for Advanced Study Berlin and the Isaac Newton Institute, Cambridge, for their kind hospitality while this work was begun. L. D. thanks the ESF QIT program and the Hungarian OTKA Grant No. 032640 for financial support. Critical comments by E. Joos, H. Wiseman, and H. D. Zeh are gratefully acknowledged.

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