Dynamical Generation of Noiseless Quantum Subsystems

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We combine dynamical decoupling and universal control methods for open quantum systems with coding procedures. By exploiting a general algebraic approach, we show how appropriate encodings of quantum states result in obtaining universal control over dynamically generated noise-protected subsystems with limited control resources. In particular, we provide a constructive scheme based on two-body Hamiltonians for performing universal quantum computation over large noiseless spaces which can be engineered in the presence of arbitrary linear quantum noise.

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Encoding of quantum information plays a vital role in strategies aimed at counteracting the effects of noise due to unwanted interactions between a quantum information processor and its environment. A chief example is quantum error correction [1], where the restriction of the initial state of the system to carefully selected subspaces of the overall state space (codes) is crucial to ensure that errors can be actively diagnosed and reversed. The identification of appropriate coded states is also the starting point of passive quantum error-avoiding approaches [2], which rely on the occurrence of specific symmetries in the system-environment interaction to obtain regions of the state space intrinsically inaccessible to noise.

An alternative solution to the issue of reliable quantum information processing in the presence of noise has been developed recently in the form of quantum error suppression techniques [3]. The latter stem from general control methods for open quantum systems [quantum bang-bang (b.b.) controls [4]], which operate by inducing suitable time scale separations between the controller and the natural dynamics of the system. Decoupling from noise is achieved by continuously undoing system-environment correlations on time scales that are short compared to the typical memory time of the environment. In contrast to the above methods, no redundant encoding is necessary for preserving and manipulating quantum information provided that the required control operations can be implemented. Such control requirements may turn out very stringent in realistic situations [5]. This raises the question of whether limited control resources, which may hinder the implementation of noise-decoupled universal quantum computation over the full system's state space, can still suffice to perform the same task over states encoded into smaller, noise-protected subsystems.

In this Letter, we investigate the usefulness of quantum coding within the decoupling framework, by examining the symmetry structure enforced on the effective dynamics by the controller. We establish a complete classification of the options available for encoding quantum information safely and for implementing universal control in a way which preserves the effect of decoupling as well as the selected coding space. The use of appropriate encodings translates into reducing the relevant control resources by allowing for either smaller or more accessible repertoires of control Hamiltonians, or for larger amounts of imperfections in the controller's operations. In particular, the need for expensive bang-bang operations is confined to maintaining noise suppression, additional manipulations on encoded subsystems becoming fully implementable via less demanding weak-strength controls.

Our analysis has several implications. First, it provides a comprehensive formalism for error suppression schemes, which incorporates previous results as a special case. Second, it further elucidates the significance of the notion of a noiseless subsystem, that has been identified as the most general route to noise-free information storage in [6] and has been argued to provide a unifying algebraic structure for noise control strategies by Zanardi [7]. Our work points out how, at variance with the case where noiseless subsystems emerge by virtue of preexisting static symmetries, additional conditions should be met to implement control over engineered noise-protected structures, thereby usefully complementing the existential results of [7]. Furthermore, the combination with coding procedures substantially expands the range of possibilities for using active decoupling methods. For a broad class of quantum information processors experiencing linear non-Markovian quantum noise, we outline a scheme where noise decoupling involves a minimal set of two bang-bang operations and, at the price of only slightly increasing the required memory resources, universal quantum logic on encoded qubits can be implemented entirely through slow tuning of two-body bilinear interactions.

Let S be a finite-dimensional open quantum system, specified as a subsystem with self-Hamiltonian H_S of a bipartite quantum system on $\mathcal{H}_S \otimes \mathcal{H}_B$, B denoting the environment. Noise is introduced in the evolution of S via a set of traceless error operators E_α in the interaction Hamiltonian, $H_{SB} = \sum_\alpha E_\alpha \otimes B_\alpha$, the B_α's being environment operators. Let $\mathcal{E}$ denote the linear space generated by the

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E_α's in the algebra $\mathcal{B}(\mathcal{H}_S)$ of operators on $\mathcal{H}_S$. Note that S and B are associated with mutually commuting operator algebras, generated by operators of the form $\{\mathcal{O}_S \otimes \mathbb{1}_B\}$ and $\{\mathbb{1}_S \otimes \mathcal{O}_B\}$, respectively, expressing the fact that S and B represent physically different degrees of freedom. For n-qubit systems, $\mathcal{H}_S \simeq \mathbb{C}^d$, $\mathcal{B}(\mathcal{H}_S) \simeq \mathrm{Mat}(d \times d, \mathbb{C})$, with $d = \dim(\mathcal{H}_S) = 2^n$.

Decoupling via bang-bang control is achieved by subjecting the system to repetitive manipulations which alternate periods of free evolution under the natural Hamiltonian with full strength/fast switching control actions able to instantaneously rotate the natural propagator U via $U \mapsto g^\dagger U g$ [3,4]. The effective dynamics takes a simple form if the unitary operators g are chosen according to a finite-order group $\mathcal{G}$ (decoupling group), $|\mathcal{G}| = \mathrm{order}(\mathcal{G}) > 1$. We identify the abstract group $\mathcal{G}$ with its image under a unitary representation μ in terms of $d \times d$ matrices. Let T_c be the time scale associated with the periodicity of the controller. In the ideal limit of arbitrarily short T_c, the dynamics of the system is modified through a quantum operation of the form [3,4,8]

$$\Pi_{\mathcal{G}}(X) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g^\dagger X g, \qquad X \in \mathcal{B}(\mathcal{H}_S). \quad (1)$$

The above group-theoretical averaging has a transparent physical interpretation. Because $\Pi_{\mathcal{G}}(X)$ commutes with every group element g, the action of the controller over times longer than the averaging period T_c only preserves the set of operators which are invariant under $\mathcal{G}$, thereby enforcing a $\mathcal{G}$ symmetrization of the evolution of S.

In order to identify suitable coding structures in $\mathcal{H}_S$, we follow an approach borrowed from quantum statistical mechanics [6,9], where abstract subsystems are associated to operator algebras with definite symmetry properties. Let $\mathbb{C}\mathcal{G}'$ denote the set of operators commuting with $\mathcal{G}$ (commutant). $\mathbb{C}\mathcal{G}'$ is a subalgebra of $\mathcal{B}(\mathcal{H}_S)$. A second algebraic structure associated with $\mathcal{G}$ is the group algebra $\mathbb{C}\mathcal{G}$, which is the (at most) $|\mathcal{G}|$-dimensional vector space spanned by complex combinations of elements in $\mathcal{G}$. $\mathbb{C}\mathcal{G}$ and $\mathbb{C}\mathcal{G}'$ are linked together by the key property of reducibility [9]. $\mathcal{G}$ acts irreducibly on $\mathcal{H}_S$ if (and only if) $\mathbb{C}\mathcal{G}' = \mathbb{C}\mathbb{1} = \{\lambda \mathbb{1}\}$, $\lambda \in \mathbb{C}$. A similar definition applies to $\mathbb{C}\mathcal{G}'$. Since the set of operators commuting with $\mathbb{C}\mathcal{G}'$ is identical to $\mathbb{C}\mathcal{G}$, the commutant is automatically reducible. Whether or not the decoupling group itself is reducible on $\mathcal{H}_S$ translates into the presence of nontrivial symmetries which can be exploited to constrain the dynamics to smaller portions of the state space.

By standard group-representation theory [9], the reduction of $\mathcal{G}$ (and $\mathbb{C}\mathcal{G}$) takes place according to the direct-sum decomposition of μ into irreducible representations (irreps) of $\mathcal{G}$, $\mu = \oplus_J n_J \mu_J$, the number of inequivalent irreps being at most $|\mathcal{G}|$. The Jth irrep, with dimension d_J, appears with multiplicity n_J, in such a way that $\sum_J n_J d_J = d$. By an appropriate change of basis, $\mathcal{H}_S$ can be made isomorphic to a direct sum over invariant spaces $\mathcal{H}_J$ of states transforming according to μ_J, $\mathcal{H}_S \simeq \oplus_J \mathcal{H}_J$. Let $\{|J; l, m\rangle | l = 1, \ldots, n_J; m = 1, \ldots, d_J\}$ denote an orthonormal basis of $\mathcal{H}_J$. Because $\mathcal{H}_J$ is made of n_J (identical) copies of a d_J-dimensional irrep space $\mathcal{D}_J$, we can further identify $|l, m\rangle \simeq |l\rangle \otimes |m\rangle$ and write $\mathcal{H}_S$ as

$$\mathcal{H}_S \simeq \oplus_J \mathcal{H}_J \simeq \oplus_J \mathcal{C}_J \otimes \mathcal{D}_J, \quad (2)$$

with $\mathcal{C}_J \simeq \mathbb{C}^{n_J}$ and $\mathcal{D}_J \simeq \mathbb{C}^{d_J}$. By construction, operators in $\mathbb{C}\mathcal{G}$ act trivially on the coordinate l specifying the irrep copy, $\delta_{l,l'}$, while operators in $\mathbb{C}\mathcal{G}'$ act trivially on the coordinate m by only mixing copies of the same irrep, $\delta_{m,m'}$. Thus, $\mathcal{H}_J$ factorizes into the tensor product of two factors $\mathcal{C}_J$ and $\mathcal{D}_J$, carrying irreps of $\mathbb{C}\mathcal{G}'$ and $\mathbb{C}\mathcal{G}$, respectively [10]. The action of $\mathbb{C}\mathcal{G}$ and $\mathbb{C}\mathcal{G}'$ over $\mathcal{H}_S$ can then be represented in the following simple form:

$$\mathbb{C}\mathcal{G} \simeq \oplus_J \mathbb{1}_{n_J} \otimes \mathrm{Mat}(d_J \times d_J, \mathbb{C}), \quad (3)$$

$$\mathbb{C}\mathcal{G}' \simeq \oplus_J \mathrm{Mat}(n_J \times n_J, \mathbb{C}) \otimes \mathbb{1}_{d_J}. \quad (4)$$

The above construction implies that we can regard each factor $\mathcal{C}_J$, $\mathcal{D}_J$ in (2) as the state space of a subsystem belonging to an effective bipartite system defined on $\mathcal{H}_J$. In analogy with a true bipartite system, $\mathcal{C}_J$ has an algebra of observables of the form $\{\mathcal{O}_{\mathcal{C}_J} \otimes \mathbb{1}_{\mathcal{D}_J}\}$, $\mathcal{O}_{\mathcal{C}_J} \in \mathrm{Mat}(n_J \times n_J, \mathbb{C})$, whereas $\mathcal{D}_J$ has an algebra of observables of the form $\{\mathbb{1} \otimes \mathcal{O}_{\mathcal{D}_J}\}\mathcal{C}_J$, $\mathcal{O}_{\mathcal{D}_J} \in \mathrm{Mat}(d_J \times d_J, \mathbb{C})$. Equations (3) and (4) are special instances of general results on operator algebras closed under Hermitian transpose [9]. In particular, the identification of noiseless subsystems supported by static symmetries can be obtained by applying a decomposition similar to (3) to the interaction algebra $\mathcal{A}$ generated by $\mathbb{1}_S$, H_S, and the E_α's [6,7]. In our setting, we are left with the freedom of exploiting any of the subsystems in (2) for encoding information. Under what conditions are such subsystems noiseless?

Let us first consider encoding in the left factors $\mathcal{C}_J$ (commutant coordinates), assuming that $n_J > 1$. Formally, in the case of one-dimensional irreps ($d_J = 1$ for some J), this kind of encoding encompasses standard noiseless subspaces, where passive protection against noise is obtained by choosing $\mathcal{C}_J$ as the singlet sector of the interaction algebra $\mathcal{A}$ [2,11]. In our case, noiselessness of the commutant degrees of freedom is guaranteed by ensuring a trivial action of the effective error generators $\Pi_{\mathcal{G}}(E_\alpha)$ over each $\mathcal{C}_J$. By observing that operators belonging to the so-called center $Z = \mathbb{C}\mathcal{G}' \cap \mathbb{C}\mathcal{G}$ are diagonal over each $\mathcal{H}_J$ [9], $Z \simeq \oplus_J \lambda_J \mathbb{1}_{n_J} \otimes \mathbb{1}_{d_J}$, $\lambda_J \in \mathbb{C}$, a necessary and sufficient condition is $\Pi_{\mathcal{G}}(E_\alpha) \in \mathbb{C}\mathcal{G}' \cap \mathbb{C}\mathcal{G}$ for every α. Note that this is generally a weaker requirement than the correctability condition $\Pi_{\mathcal{G}}(E_\alpha) = 0$ demanded in decoupling schemes without encoding [3,4]: for instance, $\mathbb{C}\mathcal{G}' \cap \mathbb{C}\mathcal{G} = \mathbb{C}\mathcal{G}$ for Abelian decouplers.

As a second coding method, we can choose the right factors $\mathcal{D}_J$ (group coordinates), provided that $d_J > 1$. In particular, such an option includes the limiting situation where $\mathcal{G}$ acts irreducibly on $\mathcal{H}_S$, in which case the decomposition (3) collapses to a single term $\mathbb{C}\mathcal{G} \simeq \mathrm{Mat}(d \times d, \mathbb{C})$ and the whole space becomes a single noiseless system [7]. This corresponds to a maximal decoupling scheme, whereby $\Pi_{\mathcal{G}}(X) = \lambda \mathbb{1}$ for every $X \in \mathcal{B}(\mathcal{H}_S)$ [4]. In the more general case where $\mathcal{G}$ is reducible, symmetrized noise generators $\Pi_{\mathcal{G}}(E_\alpha) \in \mathbb{C}\mathcal{G}'$ act trivially on factors carrying a $\mathbb{C}\mathcal{G}$-irrep. Thus, subsystems of the form $\mathcal{D}_J$ are automatically noiseless regardless of whether noise suppression over the whole state space is achieved or not. Although the effective dynamics over $\mathcal{H}_S$ is no longer unitary under these conditions, corruption of information encoded in $\mathcal{D}_J$ is fully prevented thanks to symmetry.

In addition to enabling protection against the environment, encoding may also offer improved stability against faults in the implementation of b.b. control. While imperfections of operations in $\mathcal{G}$ directly affect the group component, states that carry commutant coordinates are still unaffected as long as $\mathbb{C}\mathcal{G}'$ is preserved. Thus, encoding in subsystems of the form $\mathcal{C}_J$ is robust against imperfections of the b.b. rotations which stay in $\mathbb{C}\mathcal{G}$. This effect, which is corroborated by experience from multipulse techniques in nuclear magnetic resonance [12], will be analyzed in greater detail elsewhere.

The first step to specifying a control scheme for noiseless subsystems is to make sure that control operations are never allowed to draw states out of the selected coding space. This determines the symmetry of the Hamiltonians to be applied for control, $H \in \mathbb{C}\mathcal{G}'$ for action on $\mathcal{C}_J$ subsystems, or $H \in \mathbb{C}\mathcal{G}$ for action on $\mathcal{D}_J$ subsystems. Let $\mathcal{U}(\mathcal{C}_J)$ and $\mathcal{U}(\mathcal{D}_J)$ denote the subgroups of unitary transformations over $\mathcal{C}_J$ and $\mathcal{D}_J$, respectively. Universality results can be established by observing that, by (3) and (4), $\mathbb{C}\mathcal{G}'|_{\mathcal{C}_J} \simeq \mathrm{Mat}(n_J \times n_J, \mathbb{C})$ and, similarly, $\mathbb{C}\mathcal{G}|_{\mathcal{D}_J} \simeq \mathrm{Mat}(d_J \times d_J, \mathbb{C})$, i.e., the elements of $\mathbb{C}\mathcal{G}'$ ($\mathbb{C}\mathcal{G}$) restricted to the coding space span the whole operator algebra of the associated subsystem. Thus, by standard universality results [13], almost any pair of Hamiltonians $H_i \in \mathbb{C}\mathcal{G}'$ or $H_i \in \mathbb{C}\mathcal{G}$, $i = 1, 2$, is universal over $\mathcal{C}_J$ or $\mathcal{D}_J$, respectively. Similar results, which rely purely on symmetry arguments and therefore apply to noiseless subsystems irrespective of their static or dynamic origin, have been formally derived also in [7,11].

However, control of dynamically generated subsystems is subject to additional constraints due to the presence of the controller. Accordingly, it is crucial to specify how to apply the relevant Hamiltonians in order to achieve the desired effect. Since the averaging operation in (1) is intrinsically associated with a minimum time scale T_c [3,4], control operations should be enacted according to different criteria depending on whether the intended action is on the group or the commutant coordinates. By construction, the application of Hamiltonians in $\mathbb{C}\mathcal{G}'$ does not interfere with the controller. Thus, encoding in $\mathcal{C}_J$ has the virtue that control operations can be effected via the weak strength/slow switching scheme introduced in [3]. Eliminating the need for fast programming operations can be essential in situations where accurate frequency selection is demanded at the same time. Whenever encoding in $\mathcal{D}_J$ is chosen, slow application of arbitrary Hamiltonians produces a trivial action. Thus, the least demanding option for applying a Hamiltonian $H \in \mathbb{C}\mathcal{G}$ relies on the ability to fast-modulate H according to the weak strength/fast switching scheme outlined in [3].

If $\mathbb{C}\mathcal{G}$ is irreducible, the possibility to attain complete control over the whole $\mathcal{H}_S$ [3] is found as a special case of the above results. When $\mathcal{G}$ is reducible, reachability of arbitrary states in $\mathcal{H}_S$ necessarily occurs through control operations that steer the dynamics through different irreps of $\mathbb{C}\mathcal{G}$. The criteria for universality without encoding [3] can then be regarded in terms of a symmetry mixing which arises from either combining commutant coordinates associated with different control groups $\mathcal{G}$ or from exploiting the action on both group and commutant coordinates from a single $\mathcal{G}$. Complete controllability of noiseless subsystems does not by itself imply the potential of efficiently implementing a quantum network. This depends on the available Hamiltonians as well as on the architecture by which subsystems are configured to encode and process information. We focus on quantum computation (QC).

Let S be a quantum computer with n qubits, $\mathcal{H}_S \simeq (\mathbb{C}^2)^{\otimes n}$, and let us assume that the interaction Hamiltonian H_{SB} is linear, meaning that the error generators E_α are combinations of single-qubit Pauli operators $\sigma_a^{(i)}$, $a = x, y, z$, $i = 1, \ldots, n$. Notably, the two extreme situations of independent and collective decoherence are recovered by identifying $\{E_\alpha\} = \{\sigma_a^{(i)}\}$, $\dim(\mathcal{E}) = 3n$, and $\{E_\alpha\} = \{\sum_i \sigma_a^{(i)}\}$, $\dim(\mathcal{E}) = 3$, respectively.

Example 1: The collective spin-flips decoupling group.—Suppose that n is even and define $X_j = \sigma_x^{(j)}$, $Z_j = \sigma_z^{(j)}$, $Y_j = Z_j X_j = i\sigma_y^{(j)}$. The group of collective π rotations is the set $\mathcal{G} = \{\mathbb{1}, \otimes_{i=1}^n X_i, \otimes_{i=1}^n Y_i, \otimes_{i=1}^n Z_i\}$. $\mathcal{G}$ is an Abelian group with order $|\mathcal{G}| = 4$, generated by $\otimes_i X_i$, $\otimes_i Z_i$ and formally identical to the stabilizer of distance-two $[n, n-2, 2]$ error-correcting codes [14]. Since $\Pi_{\mathcal{G}}(\sigma_a^{(i)}) = 0$ for every a, i, decoupling according to $\mathcal{G}$ is effective at suppressing any linear interaction. A decoupling cycle is specified by a sequence of the form $[\delta - \mathcal{P}_x - \delta - \mathcal{P}_z]^2$, with $\delta = T_c/4$ a time delay and $\mathcal{P}_a$ a b.b. collective π pulse along the $\hat{a}$ axis [3]. Being Abelian, $\mathcal{G}$ has $|\mathcal{G}| = 4$ one-dimensional irreps and $\mathcal{H}_S$ decomposes as the direct sum of 4 joint eigenspaces $\mathcal{H}_J$, J representing a collective label for the generators' eigenvalues, $J = (\pm 1, \pm 1)$. Encoding into commutant factors $\mathcal{C}_J$ is the only nontrivial option. Since $\dim(\mathcal{H}_J) = n_J = 2^{n-2}$, each of the four (equivalent) subspaces is able to encode $n - 2$ logical qubits. For instance, the $\mathcal{G}$-invariant subspace $J = (1, 1)$ is spanned by the n-qubit cat states $(|x\rangle + |\textsc{not}x\rangle)/\sqrt{2}$, x denoting an even-weight binary string of length n.

In order to obtain an explicit scheme for performing universal QC on encoded qubits, the key step is to look at the available operations in $\mathbb{C}\mathcal{G}'$. As a group, $\mathbb{C}\mathcal{G}'$ has $2n - 2$ generators, two of which are also generators for $\mathcal{G}$. The $2(n-2)$ generators of $\mathbb{C}\mathcal{G}' - \mathcal{G}$ can be chosen among interactions of the form $X_i X_j$, $Z_i Z_j$, $i, j = 1, \ldots, n$. These correspond to nontrivial encoded operations. For instance, for the $J = (1,1)$ $\mathcal{G}$-invariant code spanned by the above vectors, the products $\overline{X}_j = X_1 X_{j+1}$, $\overline{Z}_j = Z_{j+1} Z_n$, $j = 1, \ldots, n-2$, act as encoded $\overline{\sigma}_x^{(j)}$ and $\overline{\sigma}_z^{(j)}$ observables, respectively. Using a Euler-angle construction, one can generate any single-qubit operation on each encoded qubit. A universal set of gates is obtained by noting that $\mathbb{C}\mathcal{G}'$ contains the Heisenberg interactions $h(i,j) = \vec{\sigma}_i \cdot \vec{\sigma}_j = X_i X_j - Y_i Y_j + Z_i Z_j$ enabling one to implement swapping between any pair of encoded qubits, i.e., $\overline{h}(i,j) = h(i+1, j+1)$. Since the square-root-of-swap gate together with one-qubit gates are a universal set [15], one can noise-tolerantly perform universal QC on $n-2$ encoded qubits by slowly turning on and off two-body interactions in parallel with the controller.

Example 2: The symmetric decoupling group.—Let $\mathcal{G} = S_n$ be the natural representation of the permutation group on $\mathcal{H}_S$, $\mu(\mathcal{P})(\otimes_{i=1}^n|\psi_i\rangle) = \otimes_{i=1}^n|\psi_{\mathcal{P}(i)}\rangle$, $\mathcal{P} \in S_n$. Starting from arbitrary linear interactions, decoupling according to S_n projects over the permutation-invariant component, leaving collective operators of the form $\sum_i \sigma_a^{(i)} \in \mathbb{C}S_n'$ as effective error generators. Thus, the controlled dynamics simulates the collective noise model [2]. Noiseless subsystems are supported only by group factors $\mathcal{D}_J$, carrying $\mathbb{C}S_n$ irreps. In the static case, such subsystems are found as commutant factors of the interaction algebra of totally symmetric operators generated by $su(2)$, $\mathcal{A} \simeq \mathbb{C}S_n'$ [7,11]. Thus, the dimensions of such coding spaces are given by the multiplicities in the Clebsch-Gordan series for $su(2)$, $\dim(\mathcal{D}_J) = (2J+1)n/[(n/2 + J + 1)!(n/2 - J)]$, $J \in \mathbf{N}/2$. Recently, a constructive scheme has been proposed for performing universal QC on qubits encoded in noiseless subsystems supported by collective noise [16]. Notably, the Heisenberg interaction alone is found to be universal on coded states. The same construction applies in our setting, with the additional constraint that the required control Hamiltonians are fast modulated at the same rate as the b.b. control within a cycle.

Example 3: The collective rotations decoupling group.—Let $\mathcal{G}$ be the continuous group generated by the Lie algebra $\mathcal{L} = su(2)$ of collective spin operators. The operators commuting with $\mathcal{G}$ belong to the group algebra $\mathbb{C}S_n$ introduced above. One can achieve decoupling according to $\mathcal{G}$ by performing the projection (1) with respect to a suitable finite-order symmetrizing group of unitaries $\mathcal{F}$ such that $\Pi_{\mathcal{G}}(E_\alpha) = \Pi_{\mathcal{F}}(E_\alpha) = 0$ for every α [11]. Noiseless subsystems can be supported by either commutant factors, in which case $\dim(C_J) = (2J+1)n!/[(n/2 + J + 1)!(n/2 - J)!]$, or by group factors, for which $\dim(\mathcal{D}_J) = 2J + 1$. If encoding in C_J subsystems is chosen, the scheme for universal QC via Heisenberg Hamiltonians described in [16] can be fully implemented according to weak/slow control.

In summary, we presented dynamical procedures for generating and controlling sectors of the state space of a generic open quantum system, which are (ideally) immune to environmental noise. The presence of nontrivial symmetries is identified as a key element common to both active error suppression and passive error avoidance methods. In spite of the mathematical resemblance, however, the two strategies are physically very different. In particular, the limit of long reservoir correlation length, which underlies the latter in the presence of collective noise, is replaced by the requirement of long reservoir correlation time in the former, which explicitly relies on the non-Markovian nature of quantum noise. The combination of decoupling and coding procedures results in a scheme for performing universal quantum computation on noise-protected subsystems which is highly appealing in terms of both the attainable encoding efficiency and the overall control resources. Our analysis suggests that appropriate use of quantum coding may allow in general for increased flexibility on ways to achieve universality and fault-tolerance in quantum computation.

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[1] P.W. Shor, Phys. Rev. A **52**, 2493 (1995); A.M. Steane, Proc. R. Soc. London A **452**, 2551 (1995); E. Knill and R. Laflamme, Phys. Rev. A **55**, 900 (1997).
[2] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997); L.M. Duan and G.C. Guo, Phys. Rev. Lett. **79**, 1953 (1997); D.A. Lidar, I.L. Chuang, and K.B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
[3] L. Viola, S. Lloyd, and E. Knill, Phys. Rev. Lett. **83**, 4888 (1999).
[4] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998); L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).
[5] Alongside Refs. [3,4], potential physical implementations of decoupling schemes are discussed by D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999); G.S. Agarwal, Phys. Rev. A **61**, 013809 (2000); C. Search and P.R. Berman, Phys. Rev. Lett. **85**, 2272 (2000).
[6] E. Knill, R. Laflamme, and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
[7] P. Zanardi, e-print quant-ph/9910016.
[8] P. Zanardi, Phys. Lett. A **258**, 77 (1999).
[9] W. Thirring, *A Course in Mathematical Physics* (Springer, New York, 1981), Vols. III and IV.
[10] By construction, the dimensions of $\mathbb{C}\mathcal{G}'$-irreps are mapped into multiplicities of $\mathbb{C}\mathcal{G}$-irreps, and vice versa.
[11] P. Zanardi, Phys. Rev. A **60**, 729 (1999).
[12] U. Haeberlen, *High Resolution NMR in Solids: Selective Averaging* (Academic Press, New York, 1976), Chap. V.
[13] S. Lloyd, Phys. Rev. Lett. **75**, 346 (1995).
[14] D. Gottesman, e-print quant-ph/9705052.
[15] G. Burkard, D. Loss, and D.P. DiVincenzo, Phys. Rev. B **59**, 2070 (1999).
[16] J. Kempe *et al.,* e-print quant-ph/0004064.