Stability of Global Monopoles Revisited

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We analyze the stability of global $O(3)$ monopoles in the infinite cutoff (or scalar mass) limit. We obtain the perturbation equations and prove that the spherically symmetric solution is classically *stable* (or *neutrally stable*) to axially symmetric, square integrable, or power-law decay perturbations. Moreover, we show that, in spite of the existence of a conserved topological charge, the energy barrier between the monopole and the vacuum is *finite* even in the limit where the cutoff is taken to infinity. This feature is specific of global monopoles and independent of the details of the scalar potential.

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Introduction.—Global monopoles have been investigated for years as possible seeds for structure formation in the Universe [1,2]. Although they appear to be ruled out by the latest cosmological data [3], their appearance in condensed matter—and other—systems and their peculiar properties make them worthy of investigation. These objects have divergent energy, due to the slow falloff of angular gradients in the fields, which has to be cut off at a certain distance R (in practice, the distance to the nearest monopole or antimonopole) and has two important consequences, in particular for cosmology. First, the evolution of a network of global monopoles is very different from that of gauged monopoles, as long-range interactions enhance annihilation to the extent of eliminating the overabundance problem altogether [1]. Second, their gravitational properties include a deficit solid angle [4], which makes them rather exotic.

The stability of global $O(3)$ monopoles has been the subject of some debate in the literature [5–7]. In this paper we try to settle the issue by (a) analyzing the axial perturbation equations in the limit where the cutoff is taken to infinity, and (b) proving that the energy barrier between the monopole and the vacuum (meaning the *extra* energy required by the monopole to reach an unstable configuration that decays to the vacuum) is *finite*. It is somewhat surprising for different topological sectors to be separated by finite energy barriers, but in this case it is a consequence of the scale invariance of gradient energy on two dimensional surfaces ($r =$ const), and therefore independent of the details of the scalar potential.

The model.—We consider the simplest model that gives rise to global monopoles, the $O(3)$ model with Lagrangian

$$L = \frac{1}{2}\,\partial_\mu\Phi^a\partial^\mu\Phi^a - \frac{1}{4}\,\lambda(|\Phi|^2 - \eta^2)^2, \quad a = 1,2,3. \tag{1}$$

Φ^a is a scalar triplet, $|\Phi| \equiv \sqrt{\Phi^a\Phi^a}$, and $\mu = 0,1,2,3$. The $O(3)$ symmetry is spontaneously broken to $O(2)$, leading to two Goldstone bosons and one scalar excitation with mass $m_s = \sqrt{2\lambda}\,\eta$. The set of ground states is the two-sphere $|\Phi| = \eta$ and, since $\pi_2(S^2) = \mathbf{Z}$, there are field configurations with nontrivial topological charge. One such configuration with unit winding is the spherically symmetric monopole,

$$\Phi^a = \eta f(r)\,\frac{x^a}{r}, \tag{2}$$

where $f(0) = 0$ and $f(r \to \infty) = 1$. Its asymptotic behavior is $f(r \to 0) \sim \alpha r$, $\alpha \approx 0.5$, and $f(r \to \infty) \sim 1 - 1/r^2 - 3/2r^4$, as can be seen from the equation of motion of $f(r)$,

$$f'' + \frac{2}{r}f' - \frac{2}{r^2}f - f(f^2 - 1) = 0. \tag{3}$$

The two parameters (η, λ) appearing in the Lagrangian can be absorbed by the rescaling $\Phi^a \to \tilde{\Phi}^a = \Phi^a/\eta$, $x^\mu \to \tilde{x}^\mu = \sqrt{\lambda\eta^2}\,x^\mu$, which amounts to choosing η as the unit of energy and the inverse scalar mass as the unit of length (up to a numerical factor). Note, however, that the energy of a configuration with nontrivial winding such as (2) is (linearly) divergent with radius, due to the slow falloff of angular gradients, and has to be cut off at $r = R$, say. Unlike η and λ, the (rescaled) cutoff is an important parameter which could affect the dynamics of solutions with nontrivial topology. Dropping tildes,

$$E = \int_0^R \frac{1}{2}\,\partial_\mu\Phi^a\partial^\mu\Phi^a + \frac{1}{4}\,(|\Phi|^2 - 1)^2. \tag{4}$$

Since the energy diverges, Derrick's theorem does not apply in this case, and in [7] it was shown that the global monopole is stable towards radial rescalings. On the other hand, the question of stability with respect to angular perturbations has led to some discussion in the literature after Goldhaber [5] pointed out that the ansatz

$$\begin{aligned} \Phi^1 &= F(r,\theta)\sin\bar{\theta}(r,\theta)\cos\varphi, \\ \Phi^2 &= F(r,\theta)\sin\bar{\theta}(r,\theta)\sin\varphi, \\ \Phi^3 &= F(r,\theta)\cos\bar{\theta}(r,\theta), \end{aligned} \tag{5}$$

which describes axially symmetric deformations of the spherical monopole (2), leads to the following expression for the energy after a change of variables $y = \ln\tan(\theta/2)$:

$$E = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dy \int_0^R dr\, \frac{1}{2}[\rho_1 + r^2 \operatorname{sech}^2(y)\rho_2],$$
$$\rho_1 = F_y^2 + F^2[\sin^2(\bar\theta) + (\bar\theta_y)^2], \quad (6)$$
$$\rho_2 = F_r^2 + F^2(\bar\theta_r)^2 + \frac{1}{2}(F^2 - 1)^2,$$

with $F_r \equiv \partial_r F$, etc. The term in brackets in ρ_1 is identical to the energy of the sine-Gordon soliton, so translational invariance in y implies that configurations with

$$F(r,y) = f(r), \quad \tan[\bar\theta(r,y)/2] = e^{y+\xi}, \quad \xi = \text{const} \quad (7)$$

have the same energy as (2) (which corresponds to $\xi = 0$). On a given $r = \text{const}$ shell, the effect of taking $\xi \to \infty$ is to concentrate the angular gradients in an arbitrarily small region around the North Pole. When the gradient energy is inside a region of size comparable to the inverse scalar mass, it is energetically favorable to "undo the knot" by reducing the modulus of the scalar field to zero and climbing over the top of the Mexican hat potential. Unwinding is estimated to occur at a critical value of ξ (say ξ_0) whose dependence with r far from the core is logarithmic, $\xi_0 \approx \ln r + \text{const}$. Numerical simulation on individual shells closer to the core gives $\xi_0 \approx a_0 + b_0 \ln r$ with slowly varying $b_0 \approx 1$ and $a_0 \approx -1.3$. Unwinding is expected if $\xi(r) > \xi_0(r)$.

Estimation of the energy barrier.—As explained in [6,7], the shift (7) creates a tension pulling the monopole core, and the apparent unwinding (which starts in the inner shells) is only a manifestation of the core's translation. In order to stop the motion of the core, we consider a hybrid configuration such that the monopole core remains unperturbed and the unwinding occurs in the outer shells. This is achieved by taking $\xi = 0$ for, say, $r < r_1$, a stringlike configuration for $r > r_2$, and some continuous interpolation in between. One such configuration [see Fig. 1a] would be

$$\hat F(r,y) = f(r), \quad \hat\xi(r) = \begin{cases} 0 & r < r_1, \\ c(1 - \frac{r_1}{r}) & r_1 < r < r_2, \\ a + b\ln(r) & r_2 < r, \end{cases} \quad (8)$$

with $a \gtrsim a_0$, $b \gtrsim b_0$, and c given by continuity of $\hat\xi(r)$.

If the energy (i.e., mass) inside r_2 is large enough there will be no appreciable motion of the monopole core (because the tension of the string is constant $\sim 4\pi$). For instance, simulations with $r_1 = 3$, $r_2 = 6$, $b = 1$ show different behavior depending on c: for $c \lesssim 2.8$ the core translates, but for $c \gtrsim 2.8$ the unwinding happens far from the core, which remains fixed. In the latter case, the decay of the string can also be understood as a monopole-antimonopole pair creation with the new monopole appearing at $r = R$ and the antimonopole appearing near $r = r_2$. The equations solved in the simulations were

$$\Box\Phi^a + \Phi^a(|\Phi|^2 - 1) + \gamma\dot\Phi^a = 0, \quad (9)$$

with a dissipative term added to make the integration faster. Simulations using different γ show no appreciable difference. The equations were integrated using cylindrical coordinates (ρ, φ, z) in a 200^2 grid using explicit Runge-Kutta method with step size control [8]. The output can be seen in Fig. 1b, where the potential energy is plotted at various times, confirming that the configuration (8) is unstable and decays to the vacuum.

We will now show that the extra energy required to reach this unstable configuration (8) from the spherical monopole configuration (2) is *finite*.

Consider the set of configurations with $F = f(r)$ and $\xi = \xi(r)$. From (6) the difference between the energy of any such configuration ($E[\xi]$) and the energy of the

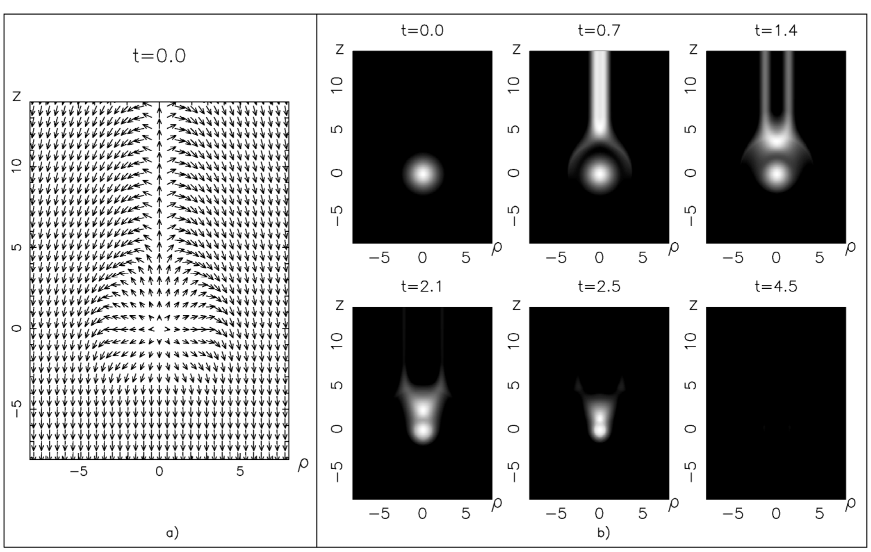

FIG. 1. (a) Plot of the scalar field Φ^a in the configuration given by Eq. (8) in the text, for $r_1 = 3$, $r_2 = 6$, $a = 0.9$, $b = 1$. The configuration is axially symmetric. (b) The result of numerical integration using (a) as initial condition. Potential energy is shown in grey scale at different times. After annihilation of the monopole-antimonopole pair near $z = 0$, the system fluctuates for some time until all the energy is radiated away (not shown; note the longer time elapsed between $t = 2.5$ and $t = 4.5$).

spherically symmetric unperturbed monopole ($E[0]$) is

$$E[\xi] - E[0] = \left[\int_0^{r_*} + \int_{r_*}^{R} \right] dr \times \left[\frac{r^2}{2} f^2(r) \xi_r^2(r) I[\xi(r)] \right], \qquad (10)$$

$$I[\xi] = \int_{-\infty}^{+\infty} dy \, \mathrm{sech}^2(y) \, \mathrm{sech}^2[y + \xi(r)].$$

$I[\xi]$ is bell shaped: from a maximum value $I[0] = 4/3$, it rapidly falls to zero for $|\xi| > \xi_* \approx 5$ as $\sim 16(|\xi| - 1)\exp(-2|\xi|)$. r_* is the radius at which $\xi(r_*) = \xi_*$. The first integral is clearly finite. The second can be estimated using the asymptotic form of $I[\xi]$ and is negligible for $\xi = a + \ln r$ even in the limit $R \to \infty$, where one gets $\sim 16\pi e^{-a} \xi_* e^{-\xi_*}$.

Moreover, there is a continuous path in configuration space connecting the configurations (2) and (8) such that $E[\xi] - E[0]$ remains finite along the whole path: first increase ξ in the outer shells $r > r_*$ using Goldhaber's deformation ($\xi_r = 0$) until ξ reaches ξ_*; then adjust the radial dependence to match (8).

Thus, we have shown that the extra energy required by the monopole (2) to go over the energy barrier and decay to the vacuum is finite even as $R \to \infty$; moreover, since the monopole energy grows with R, the ratio $(E[\hat{\xi}] - E[0])/E[0] \to 0$ as $R \to \infty$.

Stability to small perturbations with axial symmetry.—We now turn to the classical stability of (2) by considering small perturbations parametrized by

$$F = f(r) + \delta(t, r, y), \qquad (11)$$

$$\tan(\bar{\theta}/2) = [1 + \xi(t, r, y)] e^y. \qquad (12)$$

Neglecting quadratic terms gives $\sin\bar{\theta} \approx \sin\theta + \xi \sin\theta \cos\theta$, $\cos\bar{\theta} \approx \cos\theta - \xi \sin^2\theta$. Introducing $X \equiv f\xi$,

$$\Phi^1 \approx (f \sin\theta + \delta \sin\theta + X \sin\theta \cos\theta) \cos\varphi,$$
$$\Phi^2 \approx (f \sin\theta + \delta \sin\theta + X \sin\theta \cos\theta) \sin\varphi, \qquad (13)$$
$$\Phi^3 \approx (f \cos\theta + \delta \cos\theta - X \sin^2\theta),$$

which shows that the correct boundary conditions are that $\sin\theta(\delta + X\cos\theta)$ should vanish on the z axis. Note that $\delta(0)$ and $X(0)$ need not vanish. An infinitesimal translation of the monopole in the z direction corresponds to $\delta = -f_r(r)\cos\theta$, $X = f(r)/r$, and both $f_r(r)$ and $f(r)/r$ tend to ≈ 0.5 as $r \to 0$ (see Fig. 2a). There is no zero mode associated with global rotations since these have been factored out in the ansatz (5).

As usual, the perturbation equations

$$0 = r^2 \Box \delta + 2\delta + r^2[3f^2 - 1]\delta + 2X_y - 4X \tanh y,$$
$$0 = \mathrm{sech}^2 y \, r^2 \Box X + \mathrm{sech}^2 y [r^2(f^2 - 1) + 2] X + 2 \tanh y X_y - 2\delta_y \qquad (14)$$

reduce to an eigenvalue problem in ω^2 for perturbations of the form $\delta = e^{i\omega t}\hat{\delta}(r, y)$, $X = e^{i\omega t}\hat{X}(r, y)$. Eigenfunctions $\hat{\delta}, \hat{X}$ with negative eigenvalues $\omega^2 < 0$ correspond to instabilities. Dropping hats and defining $u = \tanh y$,

$$R_1\delta + \partial_u[(u^2 - 1)\partial_u \delta] - 2\partial_u[(u^2 - 1)X] = \omega^2 r^2 \delta,$$
$$R_2 X + \partial_u^2[(u^2 - 1)X] - 2\partial_u \delta = \omega^2 r^2 X, \qquad (15)$$

where R_1, R_2 are radial operators (see Fig. 2b):

$$R_i = -\partial_r(r^2 \partial_r) + V_i(r), \qquad i = 1, 2,$$
$$V_1(r) = r^2[3f^2(r) - 1] + 2, \qquad (16)$$
$$V_2(r) = r^2[f^2(r) - 1].$$

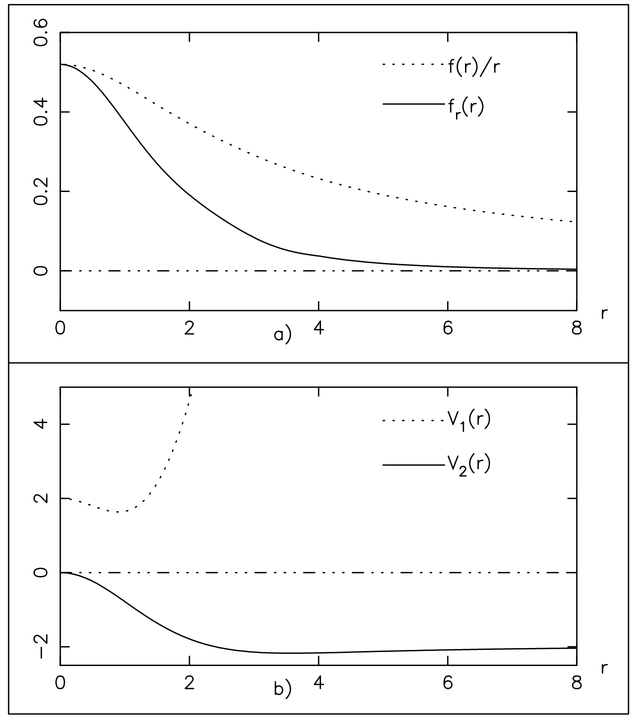

FIG. 2. The functions: (a) $f(r)/r$, $f_r(r)$; (b) $V_1(r)$, $V_2(r)$.

Using Legendre polynomials and changing variables: $\chi := \partial_u[(1 - u^2)X] = \sum \chi_l(r) P_l(u)$, $\delta = \sum \delta_l(r) P_l(u)$, the equations for different values of l decouple, giving

$$R_1 \Delta_l + x^2 \Delta_l + 2x\chi_l = \omega^2 r^2 \Delta_l, \qquad (17)$$

$$R_2 \chi_l + x^2 \chi_l + 2x\Delta_l = \omega^2 r^2 \chi_l, \qquad (18)$$

where we introduced $x = \sqrt{l(l + 1)}$ and $\Delta_l = x\delta_l$. In order to get (18) we multiplied the X equation in (15) by $(1 - u^2)$ and differentiated with respect to u, so there may be spurious solutions; in particular, $l = 0$ corresponds to angular perturbations that are singular on the z axis. These are not physical, and will be discarded. But if there is no solution of Eqs. (17) and (18) with negative ω^2 there will be no instability in the original problem (14).

Our task is to find the solution to Eqs. (17) and (18) with the minimum value of ω^2 over all admissible perturbations and all $l \geq 1$. We know one $l = 1$ solution, the translational zero mode [$\tilde{\Delta}_1 = \sqrt{2} f_r(r)$, $\tilde{\chi}_1 = -2f(r)/r$]. Goldhaber's deformation [$\chi_1 = f(r)$, $\Delta_1 = 0$] is also $l = 1$ but is *not* a solution of Eqs. (17)

and (18), and it can be shown that there are no instabilities with $\chi_1(r \to \infty) \sim$ const.

Let us first consider normalizable perturbations. Note that, for each l, Eqs. (17) and (18) can be obtained by functional variation from

$$E_l \equiv \int dr\,[r^2(\Delta_r^2 + \chi_r^2) + (V_1 + x^2)\Delta^2 + (V_2 + x^2)\chi^2 + 4x\Delta\chi] = \omega^2 \int dr\, r^2[\Delta^2 + \chi^2]. \quad (19)$$

The lowest value of ω^2 can be found minimizing E_l over normalized functions, $\int dr\, r^2[\Delta^2 + \chi^2] = 1$ and over all $l \geq 1$. However, the minimum must be in the $l = 1$ sector since, for all $l > 1$ and for given Δ, χ, $(E_l - E_1)$ is a sum of squares with positive coefficients:

$$E_l - E_1 = \int dr\,[A_{l,+}(\Delta + \chi)^2 + A_{l,-}(\Delta - \chi)^2], \quad (20)$$

where $A_{l,\pm} \equiv [x^2 - 2 \pm 2(x - \sqrt{2})]/2 > 0$ for $l > 1$.

In order to investigate the $l = 1$ sector, we rewrite E_1 using arbitrary functions $G(r)$, $H(r)$, and $K(r)$:

$$E_1 = \int_0^\infty dr\left[\left(r\Delta_r + \frac{G}{r}\Delta\right)^2 + \left(r\chi_r + \frac{H}{r}\chi\right)^2 + 2\sqrt{2}\left(\frac{\Delta}{K} + K\chi\right)^2 + \left(V_1 + 2 - \frac{2\sqrt{2}}{K^2} + G_r - \frac{G^2}{r^2}\right)\chi^2 + \left(V_2 + 2 - 2\sqrt{2}K^2 + H_r - \frac{H^2}{r^2}\right)\Delta^2\right] - G(r)\Delta^2(r)|_0^\infty + H(r)\chi^2(r)|_0^\infty. \quad (21)$$

Choosing $G(r) = -r^2\partial_r \ln\tilde{\Delta}_1$, $H(r) = -r^2\partial_r \ln\tilde{\chi}_1$, $K(r)^2 = -\tilde{\Delta}_1/\tilde{\chi}_1$, the coefficients of Δ^2 and χ^2 in (21) vanish identically by virtue of (3). For large r, G and $H \sim r$, thus, $E_1 \geq 0$ for all functions (Δ_1, χ_1) that decay faster than $1/\sqrt{r}$ as $r \to \infty$. This proves that all normalizable perturbations have $\omega^2 \geq 0$. The above argument and a host of numerical simulations strongly suggest that non-normalizable perturbations are at best compatible with $\omega^2 = 0$, as can be verified directly from the equations for perturbations that fall to zero like a power of r, but in this case we have no analytic proof.

Discussion.— In this paper we have derived and analyzed the axial perturbation equations of $O(3)$ monopoles and proved that, contrary to statements in the literature, $O(3)$ monopoles are perturbatively stable (or neutrally stable) to infinitesimal, axially symmetric, normalizable (or power-law decay) perturbations. We have also proved that the energy barrier between topological sectors is finite, irrespective of the details of the scalar potential. This feature is specific of global monopoles; global vortices in two dimensions, whose energy grows logarithmically with radius, and gauge monopoles, whose energy is finite, are separated from the vacuum by an energy barrier growing linearly with the cutoff. But the energy of global monopoles is dominated by two-dimensional (angular) gradients far from the core, and these can be deformed with no energy cost.

One would naively expect thermal fluctuations with $KT \sim E[\xi] - E[0]$ to cause the monopole to decay. As far as we know, this effect is not seen in "cosmological" numerical simulations of global monopole networks, but perhaps this is not so surprising. First, we worked in flat space. Second, the range of scales introduced by the expansion of the Universe forces drastic approximations on cosmological simulations; in particular, the sigma model approximation, which is widely used, sets the field on the vacuum manifold everywhere, so unwinding and pair creation events are not resolved by the grid. Finally, cosmological simulations do not include thermal effects, and it has been shown that full thermal simulation across the phase transition [9] gives qualitatively different results. Our results provide further evidence that a more careful analysis of global monopole networks may be required.

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