Manifestation of Superfluidity in an Evolving Bose-Einstein Condensed Gas

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We study the generation of excitations due to an "impurity" (static perturbation) placed into an oscillating Bose-Einstein condensed gas in the time-dependent trapping field. It is shown that there are two regions for the position of the local perturbation. In the first region the condensate flows around the impurity without generation of excitations demonstrating superfluid properties. In the second region the creation of excitations occurs, at least within a limited time interval, revealing destruction of superfluidity. The phenomenon can be studied by measuring the damping of condensate oscillations at different positions of the impurity.

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In studies of Bose-Einstein condensation in ultracold gases, there has been outlined a new stage associated with investigating superfluidity (SF) in such systems. This problem is of specific interest since it is a question of SF of a dilute gas. The first results in revealing a critical velocity have been obtained in [1]. At the same time in [2] the creation of a vortex is demonstrated in a sophisticated experiment with two condensates. Recently, direct creation of vortices has been observed in a rotating condensate [3].

An interesting possibility for studying SF may be realized in the observation of evolution of condensate accompanying time-dependent variations of the confining field. For the most interesting case when a parabolic potential keeps its shape, the scaling solutions found in [4–6] allow us to describe the evolution of condensate for arbitrary variations of the trap frequencies. The analysis of solutions, in particular, given below, shows that during the evolution there are space-time regions where the velocity $v(\vec{r},t)$ of condensate exceeds the local speed of sound $c(\vec{r},t)$, i.e., the local critical velocity for the SF system. Simultaneously, there exists a spatial region where, on the contrary, $v(\vec{r},t) < c(\vec{r},t)$ for any time. In the absence of external perturbations, the presence of the regions with $v(\vec{r},t) > c(\vec{r},t)$ does not result in itself in the irreversible processes, and the oscillations of condensate do not decay at a temperature $T = 0$.

However, provided an external perturbation is localized in the region where $v(\vec{r},t) > c(\vec{r},t)$ even for a limited time interval, generation of collective excitations of the condensate takes place. Naturally, this results in the decay of the oscillations of the condensate. At the same time, if the perturbation is located in the region where always $v(\vec{r},t) < c(\vec{r},t)$, the gas in the presence of SF will flow adiabatically about the local perturbation not creating excitations. Thus, measuring the decay of oscillations of the condensate under changing localization of perturbation, we can display the SF properties of a Bose-Einstein condensed gas.

As found in [4], for the isotropic 2D parabolic potential there is an exact scaling solution for arbitrary values of the gas density n and initial trap frequency ω_0 and an arbitrary dependence $\omega(t)$. This makes a choice of the trap geometry close to the cylindric one very attractive. In this case, a static perturbation can be induced by the laser beam parallel to the longitudinal cylindric axis and limited in the transverse size. Shifting the beam axis in the radial direction, one can change the localization of perturbation.

Below, we consider the evolution of the condensate in an isotropic 2D parabolic potential with the time-dependent frequency $\omega(t)$ and determine the generation of excitations under external static perturbation. The generalization for the 3D case of the cylindric symmetry and extended perturbation unchanging along the longitudinal axis can be performed easily.

Examining evolution of the 2D interacting Bose gas in the time-dependent confining potential, we can employ the general equation for the Heisenberg field operator of atoms:

$$i\hbar\frac{\partial\hat{\Psi}}{\partial t} = \left[-\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2(t)r^2}{2}\right]\hat{\Psi} + U_0\hat{\Psi}^+\hat{\Psi}\hat{\Psi}\,. \tag{1}$$

The only simplification in the equation is an assumption of the local character of the interparticle interaction.

Let us introduce a scaling parameter $b(t)$ and, correspondingly, spatial and time variables $\vec{\rho} = \vec{r}/b$, $\tau(t)$.

For the 2D case, let us represent the operator $\hat{\Psi}$ as

$$\hat{\Psi}(r,t) = \frac{1}{b}\,\hat{\chi}(\vec{\rho},\tau)\exp[i\Phi(r,t)]\,. \tag{2}$$

Substituting (2) into (1) and using the results obtained in [1], we find

$$\Phi(r,t) = mr^2\frac{1}{2\hbar b}\frac{db}{dt}\,. \tag{3}$$

Then the equation for operator $\hat{\chi}(\vec{\rho},\tau)$ reduces to

$$i\hbar\frac{\partial\hat{\chi}}{\partial\tau} = -\frac{\hbar^2}{2m}\Delta_\rho\hat{\chi} + \frac{1}{2}m\omega_0^2\rho^2\hat{\chi} + U_0\hat{\chi}^+\hat{\chi}\hat{\chi}\,. \tag{4}$$

The equation takes such form if we accept that $b(t)$ and $\tau(t)$ are determined by the equations

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$$\frac{d^2b}{dt^2} + \omega^2(t)b = \frac{\omega_0^2}{b^3}, \qquad \tau(t) = \int_0^t dt'/b^2(t'). \quad (5)$$

We put $\omega(-\infty) = \omega_0$ in Eqs. (4) and (5). Equation (5) should be solved at the initial conditions of $b(-\infty) = 1$, $\dot{b}(-\infty) = 0$.

It follows from Eq. (4) that the problem in the $\vec{\rho}$ and τ variables reduces to the solution of the equation for the static parabolic potential of the initial frequency ω_0. Finding a solution and deriving $b(t)$ and $\tau(t)$ from (5), we obtain a complete description for the space-time evolution at an arbitrary variation of $\omega(t)$.

Let us restrict our consideration with the case of zero temperature. Then the ground state in the static potential represents the condensate, and in Eq. (4) the operator $\hat{\chi}$ can be replaced by the macroscopic condensate wave function $\chi_0(\vec{\rho}, \tau)$ having a typical dependence on τ,

$$\chi_0(\vec{\rho}, \tau) = \chi_0(\vec{\rho})e^{-i\mu\tau/\hbar}. \quad (6)$$

Here, μ is the initial chemical potential and $\chi_0(\vec{\rho})$ is the solution of the equation

$$-\frac{\hbar^2}{2m}\Delta_\rho\chi_0 + \left[-\mu + \frac{1}{2}m\omega_0^2\rho^2 + U_0\chi_0^2\right]\chi_0 = 0. \quad (7)$$

To describe excitations in the system, we introduce operator $\hat{\chi}'(\vec{\rho}, \tau)$. Let us write the initial operator $\hat{\chi}$ as $\hat{\chi}(\vec{\rho}, \tau) = [\chi_0(\vec{\rho}) + \hat{\chi}'(\vec{\rho}, \tau)]e^{-i\mu\tau/\hbar}$ and substitute it into (4). Then we find the following for the equation linearized in $\hat{\chi}'(\vec{\rho}, \tau)$:

$$i\hbar\frac{\partial\hat{\chi}'}{\partial\tau} = \left[-\frac{\hbar^2}{2m}\Delta_\rho + \frac{1}{2}m\omega_0^2\rho^2 - \mu\right]\hat{\chi}' + 2U_0\chi_0^2\hat{\chi}' + U_0\chi_0^2\hat{\chi}'^+. \quad (8)$$

This equation determines excitations in the coordinates of $(\vec{\rho}, \tau)$, and we can consider any problem in this reference frame. However, keeping in mind the solution of the problem on creating excitations under a static perturbation, it is more natural to investigate it in the laboratory reference frame.

Let us rewrite Eq. (8) introducing new variables $\vec{r}$ and t. For this purpose, we use the relations

$$\frac{\partial\hat{\chi}'}{\partial\tau} = b^2\frac{\partial\hat{\chi}'}{\partial t} + \vec{r}b\dot{b}\frac{\partial\hat{\chi}'}{\partial\vec{r}}, \qquad \Delta_\rho\hat{\chi}' = b^2\Delta_r\hat{\chi}'.$$

We assume that the density of a gas is sufficiently large and $\mu \gg \hbar\omega_0$, i.e., the Thomas-Fermi approximation is valid. Then we find the condensate density from Eq. (7) [see (2)]:

$$n_0(\vec{r}, t) = \frac{1}{b^2(t)}\chi_0^2, \qquad \chi_0^2 = \frac{\mu}{U_0}\left(1 - \frac{r^2}{R^2(t)}\right). \quad (9)$$

where $R(t) = R_0 b(t), R_0 = \sqrt{2\mu/m\omega_0^2}$.

As a result, in the laboratory reference frame Eq. (8) reads

$$i\hbar\frac{\partial\hat{\chi}'}{\partial t} = -\frac{\hbar^2}{2m}\Delta_r\hat{\chi}' + g\hat{\chi}' + g\hat{\chi}'^+ - i\hbar\vec{v}\nabla_r\hat{\chi}'. \quad (10)$$

Here

$$\vec{v} = \vec{r}\frac{\dot{b}}{b}, \qquad g = U_0 n_0(\vec{r}, t). \quad (11)$$

It is easy to see that the quantity $\vec{v}$ is a local velocity of the condensate. In fact, the total condensate wave function equals

$$\Psi_0(r, t) = \frac{1}{b}\chi_0\exp\left[i\Phi(r, t) - \frac{i\mu\tau(t)}{\hbar}\right]. \quad (12)$$

Taking into account (3) and determining phase gradient, we arrive at expression (11) for the velocity. Special attention should be paid to the appearance of the Doppler $\vec{v}\hat{p}\hat{\chi}'$ term in Eq. (10), $\hat{p} = (-i\hbar\nabla)$ being the operator of momentum.

Note that, according to (9), the density$n_0(\vec{r}, t)$ is proportional to $b^{-2}(t)$. Correspondingly, the correlation length of $\xi \sim n_0^{-1/2}$ changes in time as $\xi = \xi_0 b(t)$, where $\xi_0 = \hbar(2mU_0n_0(0,0))^{-1/2}$. The size of the system $R(t)$ changes in the same manner. Thus a ratio of these quantities $\xi(t)/R(t) = \xi_0/R_0$ conserves for an arbitrary scale of evolution.

The inequality $\xi_0/R_0 \ll 1$ is crucial for the Thomas-Fermi approximation. So, if this approximation is valid at the initial time, it remains valid for an arbitrary time. Since $\xi(t)/R(t) \ll 1$, we have a quasiuniform problem at any time, considering excitations of the $\xi(t) \leq \lambda \ll R(t)$ wavelengths. The scale of the uniform regions, according to Eq. (10), changes with the typical time of $t_{\mathrm{eff}} \approx b(t)/\dot{b}(t)$. The well-defined excitations satisfy the inequality of $\omega t_{\mathrm{eff}} \gg 1$, i.e., satisfy the quasistationary condition.

Let us consider the case of the fast transition from frequency ω_0 to $\omega_1 = \omega_0/\beta$, $\beta > 1$. In this case, the solution of Eq. (5) yields

$$b^2(t) = \frac{1}{2}(\beta^2 + 1) - \frac{1}{2}(\beta^2 - 1)\cos 2\omega_1 t. \quad (13)$$

Hence,

$$\vec{v} = \vec{r}\frac{\omega_1}{2b^2}(\beta^2 - 1)\sin 2\omega_1 t. \quad (14)$$

As we will see below, the excitations of $\lambda \sim \xi$ give the main contribution into the damping. The energy of these excitations is of the order of $U_0 n_0(0, t) = \mu/b^2(t)$ and, correspondingly,

$$\omega t_{\mathrm{eff}} \approx \frac{2\mu}{\hbar\omega_0}\frac{\beta}{(\beta^2 - 1)\sin 2\omega_1 t}.$$

It is clear that the $\omega t_{\mathrm{eff}} \gg 1$ condition holds if $2\mu/\hbar\omega_0 \gg \beta$. The last inequality can easily be fulfilled. Thus, we can seek for the solution of Eq. (10) within the quasiuniform and quasistationary approximation.

Let us introduce a typical representation for operator $\hat{\chi}'$,

$$\hat{\chi}'/b = \sum_{\vec{k}} \exp(i\vec{k}\vec{r})\hat{a}_{\vec{k}}, \tag{15}$$

where $\hat{a}_{\vec{k}}$ is the annihilation operator of a particle. We employ the known Bogoliubov transformation, e.g., [7], for the operator $\hat{a}_{\vec{k}}$:

$$\hat{a}_{\vec{k}} = u_{\vec{k}}\hat{b}_{\vec{k}} - v_{\vec{k}}\hat{b}^+_{-\vec{k}}, \tag{16}$$

where $\hat{b}^+_{\vec{k}}$ and $\hat{b}_{\vec{k}}$ are the creation and annihilation operators of collective Bose excitations. As a result, the solution of Eq. (10) gives the excitation spectrum,

$$\varepsilon_{\vec{k}} = \hbar\vec{k}\vec{v} + \tilde{\varepsilon}_{\vec{k}}, \qquad \tilde{\varepsilon}_{\vec{k}} = \sqrt{E^2_{\vec{k}} - g^2}, \qquad E_{\vec{k}} = \frac{\hbar^2\vec{k}^2}{2m} + g. \tag{17}$$

The expression for the coefficients of the transformation is given by

$$u^2_{\vec{k}} = \frac{1}{2}\left(\frac{E_{\vec{k}}}{\tilde{\varepsilon}_{\vec{k}}} + 1\right), \qquad v^2_{\vec{k}} = \frac{1}{2}\left(\frac{E_{\vec{k}}}{\tilde{\varepsilon}_{\vec{k}}} - 1\right). \tag{18}$$

Let an external perturbation of a size of about $d \ll R_0$ and localized near $\vec{r} = \vec{r}_p$ be described by an effective interaction $B(\vec{r} - \vec{r}_p)$. The Hamiltonian of the interaction of the gas with the perturbation has the form [see Eq. (2)]

$$H_{\text{int}} = \frac{1}{b^2}\int d^2r\, \hat{\chi}^+(\vec{r},t)B(\vec{r} - \vec{r}_p)\hat{\chi}(\vec{r},t).$$

Selecting terms corresponding to the single-particle excitations, we have

$$H_{\text{int}} = \frac{1}{b^2}\int d^2r[\chi_0(\vec{r},t)B(\vec{r} - \vec{r}_p)\hat{\chi}'(\vec{r},t) + \text{H.c.}]. \tag{19}$$

Let us transform this Hamiltonian, using (15), (16), and the value of $\chi_0 = b\sqrt{n_0(\vec{r}_p,t)}$:

$$H_{\text{int}} = \sqrt{n_0(\vec{r}_p,t)}\sum_{\vec{k}} B_{\vec{k}}e^{i\vec{k}\vec{r}_p}[u_{\vec{k}} - v_{\vec{k}}]\,(\hat{b}_{\vec{k}} + \hat{b}^+_{-\vec{k}}). \tag{20}$$

If we take into account Eq. (17), the probability per unit time of creating excitations equals ($T = 0$)

$$W = \frac{2\pi}{\hbar}n_0(\vec{r}_p,t)\int\frac{d^2k}{(2\pi)^2}(B_k)^2[u_k - v_k]^2\delta(\tilde{\varepsilon}_k + \hbar\vec{k}\vec{v}). \tag{21}$$

In accordance with Eq. (18), $(u_k - v_k)^2 = \hbar^2k^2/2m\tilde{\varepsilon}_k$. Integration over the angles in Eq. (21) yields

$$\int d\varphi\,\delta(\hbar kv\cos\varphi + \tilde{\varepsilon}_k) = \frac{2\theta(1 - (\tilde{\varepsilon}_k/\hbar kv))}{\hbar kv\sqrt{1 - (\tilde{\varepsilon}_k/\hbar kv)^2}},$$

where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. Then

$$W = \frac{n_0(\vec{r}_p,t)}{2\pi vm}\int dk|B_k|^2\frac{k^2}{\tilde{\varepsilon}_k}\frac{\theta(1 - (\tilde{\varepsilon}_k/\hbar kv))}{\sqrt{1 - (\tilde{\varepsilon}_k/\hbar kv)^2}}. \tag{22}$$

As follows from this expression, creation of an excitation with the wave vector k occurs provided that $v > \tilde{\varepsilon}_k/\hbar k$, i.e., provided that the familiar Landau criterium holds. The generation of excitations is completely absent if $v(\vec{r}_p,t) < c(\vec{r}_p,t)$.

The local velocity of sound equals [see (17)]

$$c(\vec{r},t) = \sqrt{\frac{g(\vec{r},t)}{m}} = \frac{c_0}{b(t)}\left(1 - \frac{r^2}{R^2(t)}\right)^{1/2},$$

where $c_0 = \sqrt{\mu/m}$. Taking into account the local magnitude of the gas velocity (14) and expression (13), we have

$$\alpha = \frac{v^2(\vec{r}_p,t)}{c^2(\vec{r}_p,t)} = \frac{2\eta^2(b^2 - 1)(\beta^2 - b^2)}{\beta^2(b^2 - \eta^2)}\theta(b - \eta), \tag{23}$$

where $\eta = r_p/R_0$. For $1 < \eta < \beta$, the dependence of α on $b^2(t)$ is a monotonically decreasing function with the inevitably existing region where the generation of excitations ($\alpha > 1$) takes place. If $1 < \eta \ll \beta$, then $b_{\min} = \eta$, $b_{\max} = \beta(1 - 1/4\eta^2)$.

If $\eta < 1$, the value α as a function of b^2 has a maximum. Calculating α_m at this point, we can verify that the generation of excitations does not appear in the whole region of $r_p < R_0$ if $\beta < \sqrt{2}$. At the same time, the generation is absent at $r_p < R_0/\sqrt{2}$ for an arbitrary value of β.

Suppose that $r_p > R_0$, and let us calculate an integral in Eq. (22). The values of k are limited by the condition which follows from the equality $\tilde{\varepsilon}_k = \hbar kv$,

$$k \leq k_*, \qquad k_* = \frac{2m}{\hbar}\sqrt{v^2 - c^2}. \tag{24}$$

In the 2D case, the effective vertex B_k in the interaction Hamiltonian (20) at $k_*d < 1$ can be represented as

$$B_k = \frac{B^{(0)}_k}{1 + (mB^{(0)}_k/\pi\hbar^2)\ln(1/kd)}, \tag{25}$$

where $B^{(0)}_k$ is the Fourier transform of the bare potential, d being a length of the order of the size of perturbation potential in the 2D case. If this value is large enough, we have

$$B_k = \frac{\pi\hbar^2}{m\ln(1/kd)}, \tag{26}$$

i.e., it has only a weak logarithmic dependence on k. In the opposite case $k_*d > 1$ Eqs. (23) and (26) are not valid and the calculation of the transition amplitude requires a knowledge of the concrete form of the interaction potential. The most contribution into the integral (22) is given by the interval adjoined the upper limit k_*. We can approximately take a factor of $|B_k|^2$ out of the integral, putting k_* into the argument of the logarithm. Next, integral (22) can be calculated exactly as an integral over $x = (\tilde{\varepsilon}_k/\hbar v k)$ within the limits $(c/v, 1)$:

$$W = \frac{mn_0(\vec{r}_p, t)\,(B_*)^2}{\hbar^3}\left[1 - \frac{2}{\pi}\arcsin\frac{c}{v}\right]\theta\left(1 - \frac{c}{v}\right), \tag{27}$$

where $B_* = B_{k_*}$. Let us determine the total number of excitations created for a period of the condensate oscillations in the case of $1 < \eta \ll \beta$:

$$I = 2\int_{t_1}^{t_2} dt\, W\,. \tag{28}$$

The time t_1 determines the moment when the boundary of the gas front reaches the point r_p. This value is found from the condition $r_p = b(t_1)R_0$. It follows from Eq. (13) that $b^2(t_1) \approx 1 + (\omega_0 t_1)^2$ and $t_1 \approx \sqrt{\eta^2 - 1}/\omega_0 \ll 1/\omega_1$. The time t_2, found from the condition $\alpha = 1$ (23), equals $t_2 \approx [\pi/2 - 1/(\sqrt{2}\,\eta)]/\omega_1$. Substituting Eq. (9) for the density into (27), one can see that the integral over time is dominated by the lower limit. From the physical point of view, this is associated with the fact that the condensate density at point r_p has a maximum during evolution just for $t \sim t_1$. It follows from (23) that $c/v \le 1/(\sqrt{2}\,\eta)$. Under this condition, the factor in the square brackets in Eq. (27) is of the order of unity. After integration, we find

$$I \approx \frac{1}{\hbar^3}\, mn_0(0,0)\,|B_*|^2\,\frac{1}{\eta\,\omega_0}\,. \tag{29}$$

Substituting (26) into (29) and introducing the total number of particles $N = \frac{1}{2}\,\pi R_0^2 n_0(0,0)$ instead of $n_0(0,0)$, we find

$$I \approx \frac{\pi\hbar\omega_0}{\eta\mu}\,\frac{N}{\ln^2(1/k_* d)}, \qquad \eta = \frac{r_p}{R_0}\,. \tag{30}$$

In the present case we have put $\hbar\omega_0/\mu \ll 1$. This means that the number of excitations I and also the number of particles escaped from the condensate for the oscillation period are small compared with N.

The total energy of the excitations generated during one oscillation period can be estimated as $\mu\eta^2 I$. Since the energy of the oscillating condensate is about μN, the relative energy loss for a period at $\eta \ge 1$ and $\beta \gg 1$ approximately equals $\eta^2 I/N$.

Treating a realistic problem in the quasi-2D case corresponding to the cylindric symmetry of the field configuration with the transverse parabolic potential and perturbation independent of the longitudinal coordinates, we arrive at the result of (30) with N equal to the total 3D number of particles. The cylindric symmetry of perturbation results in creating collective excitations of $k_z = 0$ alone. The problem of scattering proves to be purely two dimensional and we arrive at Eqs. (26) and (30). The quantities $B_k^{(0)}$ and B_k remain as 2D Fourier components. Note that, if the laser beam parallel to the cylindric axis is used as a static perturbation (possibly, with the light sheets at the edges), an influence of the regions where the beam enters and leaves the condensate is unessential.

In conclusion, we have investigated the generation of excitations in the oscillating condensate in a time-dependent parabolic trap at the presence of a static local perturbation. We have revealed the existence of two space regions for the position of an "impurity." In the first region of $r_p < R_0/\sqrt{2}$, the generation of excitations is absent at any scale of oscillations. In the second one of $r_p > R_0$, the generation is realized inevitably for an arbitrary set of parameters and we have found the generation rate. The results for the first region are a direct consequence of SF of condensate. For any position in the second region, there is a finite time interval when the local velocity of the condensate proves to be larger than the local velocity of sound, entailing violation of SF. Measurements of the damping of the condensate oscillations at different positions of the perturbation opens a possibility to study the manifestation of SF in the evolving condensate.

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