Potential Energy in a Three-Dimensional Vibrated Granular Medium Measured by NMR Imaging

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Fast NMR imaging was used to measure the density profile of a three-dimensional granular medium fluidized by vertical vibrations of the container. For container acceleration much larger than gravity, the rise in center of mass of the granular medium is found to scale as v_0^α/N_ℓ^β with $\alpha = 1.0 \pm 0.2$ and $\beta = 0.5 \pm 0.1$, where v_0 is the vibration velocity, and N_ℓ is the number of layers of grains in the container. This value for α is significantly less than found previously for experiments and simulations in one dimension ($\alpha = 2$) and two dimensions ($\alpha = 1.3-1.5$).

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One technologically important way to fluidize a granular medium such as sand is to vibrate the walls containing the system. Basic issues include the scaling of the kinetic and the potential energy of the grains with parameters such as the frequency and the amplitude of the wall vibration, and the number of layers of grains. These scaling relations are not yet fully understood for multidimensional systems, despite progress in simulations [1], experiments [2], and theory [3,4].

We consider a granular system in gravity excited by periodic vertical motion of the container $z(t) = z_0 \cos(\omega t)$. When the dimensionless acceleration $\Gamma = z_0\omega^2/g$ is larger than unity, the granular system enters a state of internal motion determined in part by Γ, the dimensionless frequency $\overline{\omega} = \omega(d/g)^{1/2}$, and the aspect ratio of the system. Here, g is the acceleration of gravity and d is the grain diameter, so $(d/g)^{1/2}$ is the time for a grain to fall through its radius [1]. For $\overline{\omega} \ll 1$, the container vibration induces large-scale structures in the granular medium such as density and surface waves. Surface structures are also observed for larger $\overline{\omega} \approx 1$ in very low aspect-ratio systems that permit small-wave-number symmetry breaking [5,6]. Conversely, for $\overline{\omega} \gg 1$ the container vibration is fast compared with the grain-scale motion. In this limit, the granular medium can assume a fluidized state analogous to a gas or liquid, and possibly amenable to theoretical methods developed for such systems [2,3,7–9]. The major distinguishing characteristic of a *granular* fluid is the continuous energy loss to inelastic collisions, which if sufficiently great can destroy the statistical uniformity of the system via inelastic collapse [10].

In this paper we report an experimental study of a three-dimensional granular system in the fluidized regime $\overline{\omega} \approx 4$, $\Gamma \leq 14$. Past numerical and experimental work has concentrated on one- and two-dimensional systems [1,2,11–14], due in part to limited computational resources and the lack of experimental methods suitable for examining the interiors of dense, flowing three-dimensional media. Three-dimensional vibrated granular systems have been studied previously at lower $\overline{\omega}$ by observing the height rise of a lid [15,16] or by using an inductive pickup [16].

Recently, nuclear magnetic resonance (NMR) has emerged as a powerful noninvasive method for studying three-dimensional granular media [17–19]. In the experiments reported here, NMR imaging was used to measure the density distribution during continuous vibration of the container. Incoherent grain motion limited image acquisition time to approximately 1 ms, too short to permit ordinary two-dimensional imaging. Therefore one-dimensional images were acquired showing the density projected onto the vertical axis $\rho(z)$. We find that $\rho(z)$ deviates strongly from the exponential form e^{-Cz} expected for a dilute isothermal gas in gravity [3]. We focus here on the scaling of the gravitational potential energy, as measured by the center-of-mass height, with Γ and the number of grain layers N_ℓ. In the present three-dimensional experiments, the scaling is found to deviate even more strongly from ideal-gas predictions than was found in earlier two-dimensional studies [2,13].

The granular medium used for these experiments was composed of mustard seeds, of mean diameter $d = 1.8$ mm and mass 4.0 mg. Under magnification, the seeds appeared roughly spherical with typical eccentricity $\pm 15\%$. Seeds are used because their oily centers give strong, long-lived NMR signals [17,18]. The seeds were held in a cylindrical container formed by a vertical glass tube of inside diameter 0.8 cm with a flat Teflon bottom wall. The container was sufficiently tall to prevent collisions between the seeds and the top wall. It was determined visually that the number of seeds required to complete a monolayer in this container was 18 ± 2. The number of layers N_ℓ was determined by counting the seeds and dividing by the monolayer number. The experiments were carried out under ambient atmospheric pressure.

A vertical Fiberglas tube was used to support the container at the center of the NMR probe, which was installed in a vertical-bore superconducting solenoid. The static field was set to $B_0 = 1.00$ T to reduce susceptibility contrast effects in these highly inhomogeneous samples. The

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lower end of the Fiberglas tube was mounted to a loudspeaker, driven by a function generator and power amplifier, to provide vertical vibrations of the container. The waveform and amplitude of the vertical motion were measured by a micromachined accelerometer (Analog Devices ADXL50) mounted to the Fiberglas tube, close to the NMR probe. This accelerometer was initially calibrated against a more accurate unit, and its output was digitized and fitted. Data were taken at two values of the vibration frequency $\omega/2\pi = 50$ and 40 Hz. Acceleration values were set to within $\pm 0.05g$.

One-dimensional NMR images were obtained using a spin-echo sequence. Gradient pulses of fixed amplitude 61.5 mT/m were used to encode vertical position information and an echo time TE = 1.0 ms was selected. Thus, the entire imaging sequence was complete within 1.5 ms. These parameters were chosen to achieve adequate spatial resolution $\delta z = 800$ μm within a time much smaller than the vibration period while holding signal loss due to incoherent grain motion to an acceptably small level. This loss, measured by the echo size, was greater for smaller samples and larger vibration amplitudes. For a midsize sample with $N_\ell = 3$, the loss was approximately 10% at the highest amplitude used.

The NMR acquisition was triggered synchronously with the sample vibration. For each value of N_ℓ and Γ, data for 100 different trigger delays spread uniformly over the vibration period were averaged together. Thus, the data presented here represent the sample density *averaged over the vibration cycle.* To remove effects of rf-field inhomogeneity the data were normalized to data obtained for a stationary water-filled tube.

Figure 1 shows the vertical density profile as a function of Γ for one of the (N_ℓ, ω) points used. With no vibration ($\Gamma = 0$) the profile is deeply corrugated, reflecting the layering of grains in the stationary chamber. As the vibration amplitude is increased, the layer structure is smoothed out. This is due to the relative motion of the grains and not simply the vertical motion of the chamber, as the latter has an amplitude z_0 that is less than the grain diameter for most of our data. In addition, the typical grain motion is less than z_0 at high Γ. Because $\overline{\omega} > 1$, the grains do not fall fast enough to follow the container motion on its downwards stroke.

Even at the highest Γ used, it is clear from Fig. 1 that the density profile deviates strongly from the exponential form expected for an isothermal ideal gas. Although the density tail at the upper (free) surface might reasonably be fit to an exponential, the density levels off and then decreases as the lower surface is approached. This agrees qualitatively with features observed in experiments and simulations for two-dimensional systems [2,13]. The shape of the lower surface depends somewhat on our choice to present $\rho(z)$ averaged over the vibration cycle, but the large deviations from an exponential profile that occur over the central portion of the sample are qualitatively the same for averaged and unaveraged data.

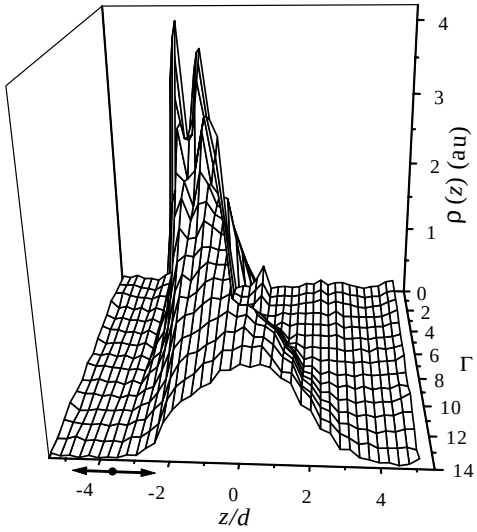

FIG. 1. Density profiles $\rho(z)$ for $N_\ell = 2.8$ grain layers vibrated at 40 Hz ($\overline{\omega} = 3.41$). The acceleration of the vibration relative to gravity Γ was varied from 0 to 14. The height coordinate z is normalized by the particle diameter d. The zero for the z/d axis is arbitrary, as are the units of the density axis. The doubled-headed arrow shows the peak-to-peak container excursion $\pm z_0$ for the largest Γ used. With no vibration ($\Gamma = 0$), the grains are organized in distinct layers which appear as peaks in $\rho(z)$. At high values of Γ, the layer structure is eradicated and a smooth density profile is observed.

To quantify the gravitational potential energy of the system, we have computed the center-of-mass height $h_{\text{c.m.}} = [\int z\rho(z)\,dz]/[\int \rho(z)\,dz]$. For each N_ℓ, the limits of integration were kept fixed as Γ was varied. Then the rise in center-of-mass height $\Delta h_{\text{c.m.}}$ was computed by subtracting the $\Gamma = 0$ value of $h_{\text{c.m.}}$ for the same N_ℓ. With this procedure, $\Delta h_{\text{c.m.}}$ gives the mean vertical displacement of the grains due to vibration *relative to the time-averaged container position.* The normalized center-of-mass rise data for 50 Hz vibration frequency are shown in Fig. 2.

Small systematic dips can be noted in Fig. 2 for specific values of N_ℓ (e.g., 5.6). These do not reflect special states of the fluidized ($\Gamma \gg 1$) state for these N_ℓ values, but rather variations in the static ($\Gamma = 0$) packing of the particles which affect the subtraction used to compute $\Delta h_{\text{c.m.}}$. These dips could be removed from Figs. 2 and 3 by fitting $\Delta h_{\text{c.m.}}$ to determine the $\Gamma = 0$ limit of the *fluidized state.* We prefer to display the data subtracting the *measured* $\Gamma = 0$ $h_{\text{c.m.}}$ to avoid prejudging the functional form.

To ascertain the functional form, we have plotted the $\Delta h_{\text{c.m.}}$ data scaled by various powers of z_0 and ω. Figure 3 shows the best data collapse obtained in this way, which is found to be $\Delta h_{\text{c.m.}}/\overline{v}_0$. Here $\overline{v}_0 = v_0/(dg)^{1/2}$ is the velocity amplitude of the vibration $v_0 = \omega z_0$, normalized by the velocity $(dg)^{1/2}$ of a grain after it falls from rest through its radius. In this plot, the data for low acceleration $\Gamma \leq 6$ do not collapse while the data for high acceleration $\Gamma > 6$ all collapse onto a single function of N_ℓ. The collapse is significantly better for $\Delta h_{\text{c.m.}}/v_0$ than it is for $\Delta h_{\text{c.m.}}/v_0^{0.5}$ or $\Delta h_{\text{c.m.}}/v_0^{1.5}$, from which we conclude that

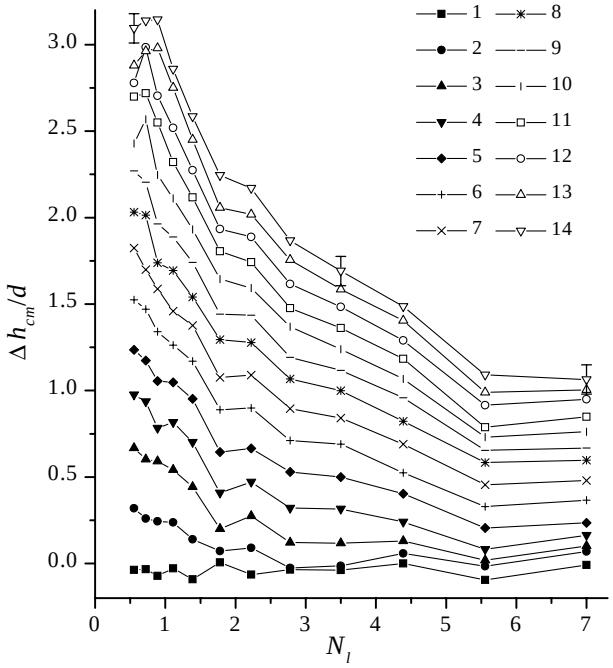

FIG. 2. Rise in the center of mass of the grains relative to the average container height, as a function of number of layers N_ℓ and acceleration Γ at vibration frequency 50 Hz ($\overline{\omega} = 4.26$). The center-of-mass rise $\Delta h_{\text{c.m.}}$ is normalized by the grain diameter d. The legend gives the Γ value for each set of symbols. Error bars typical of the entire data set are shown on some symbols.

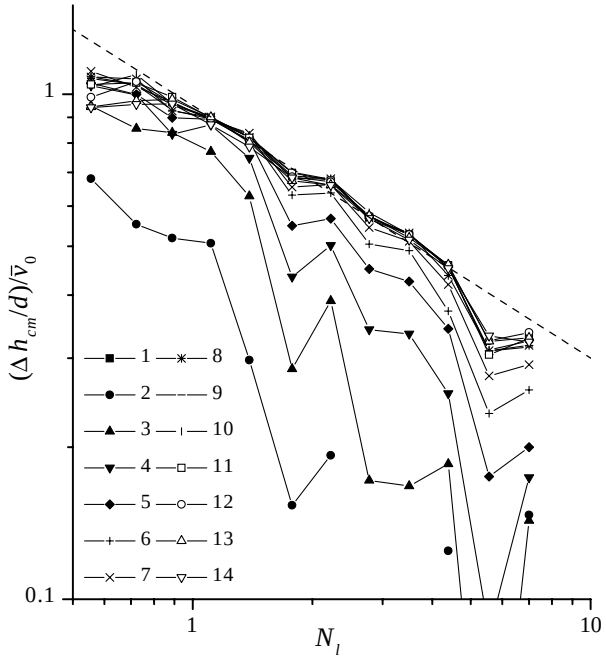

FIG. 3. The data of Fig. 2, divided by the dimensionless vibration velocity $\overline{v}_0$ and plotted on log axes. The symbols are the same as in Fig. 2. The eight curves for $\Gamma = 7-14$ lie nearly on top of one another, implying that $\Delta h_{\text{c.m.}}$ scales as $\overline{v}_0^{\alpha}$ with $\alpha \approx 1$. The dashed line is the function $(0.95)/N_\ell^{1/2}$.

$\Delta h_{\text{c.m.}}$ scales as v_0^{α} with $\alpha = 1.0 \pm 0.2$. As shown by the dashed line in Fig. 3, the variation of $\Delta h_{\text{c.m.}}$ with N_ℓ at high Γ is approximately $N_\ell^{-1/2}$. From plots such as this for a range of exponents, we ascertain that $\Delta h_{\text{c.m.}}/v_0 \propto N_\ell^{-\beta}$, with $\beta = 0.5 \pm 0.1$.

The scaling with ω cannot be determined from a data set at fixed vibration frequency as shown in Fig. 3. When the data for vibration frequency 40 Hz are added to the scaling plot, it appears that the velocity ωz_0 is a better scaling variable than is the amplitude z_0 or the acceleration $\omega^2 z_0$. However, due to the limited variation of ω this conclusion is not firmly established by our experiments.

The simplest model for a vibrofluidized granular medium is the elastic hard-sphere gas. In this model the mean grain velocity is proportional to the container vibration velocity, and the potential energy is proportional to the kinetic energy; hence, $\Delta h_{\text{c.m.}} \propto v_0^2$. This velocity scaling has been observed in experiments and simulations of *one-dimensional* granular systems (columns of nearly elastic beads) [12]. *Two-dimensional* experiments [2] and simulations [13] found $\Delta h_{\text{c.m.}} \propto v_0^{\alpha}$ with significantly smaller $\alpha = 1.3-1.5$, a result that is only partially understood [1,4]. From a phenomenological viewpoint, our result $\alpha \approx 1$ in three dimensions appears to continue the trend with increasing dimensionality away from elastic kinetic-theory predictions.

For a less superficial understanding, we must consider the reasons for deviations from ideal-gas theory that have been proposed. The steady state of the granular system is determined by a balance between work done on the grains by collisions with the vibrating container bottom and energy lost due to inelastic collisions between the grains [1]. Other loss mechanisms discussed in the literature but not considered here include collisions with the side walls and viscous damping due to interstitial gas [3].

We first discuss the power P_i that is fed into the translational degrees of freedom by collisions with the vibrating bottom wall. An important dimensionless parameter is the ratio of the mean random grain velocity v_g to the characteristic wall vibration velocity v_0. For nearly elastic systems (restitution coefficient close to one) with a small number of layers N_ℓ, it is possible to have $v_g/v_0 \gtrsim 1$. Conversely, less elastic systems (including our experimental system) generally have $v_g/v_0 \ll 1$ in the uniformly fluidized state at large $\overline{\omega}$.

Collisions between the grains and the vibrating bottom wall always transfer upward momentum to the grains, but do positive work on the grains only when the wall is moving upward. When $v_g/v_0 \ll 1$ the grains are unable to catch up with the wall on its downward stroke. In this case, equating the momentum transfer per unit time with the sample weight $\propto N_\ell$ gives the rigorous result $P_i \propto v_0 N_\ell$ [20]. The same result is obtained for an asymmetric sawtooth vibration waveform (infinitely fast downward stroke) for any value of v_g/v_0. Thus the symmetry of the drive waveform is significant only when $v_g/v_0 \gtrsim 1$ [20]. For a symmetric waveform, it has been suggested and verified for two-dimensional simulations that $P_i \propto v_0 N_\ell f(v_g/v_0)$,

where $f(0) = 1$ and $f(x)$ is a decreasing function of x [1]. McNamara and Luding [1] fit their simulation data for symmetric drive waveforms to $f(x) = e^{-Dx}$. We suggest their data may also be approximately represented by $f(x) = 1/(1 + x)$. For large v_g/v_0, this gives $P_i \propto v_0^2 N_\ell / v_g$, a result that has been derived directly from kinetic theory for symmetric drive waveforms in the nearly elastic limit [3].

For a dilute, nearly elastic, isothermal system, kinetic theory predicts that the density decays exponentially with height and the power lost due to collisions $P_c \propto N_\ell^2 v_g$ [3]. Here isothermal denotes uniform granular temperature $T \propto v_g^2$. In this limit, potential energy is strictly proportional to kinetic energy, $\Delta h_{\mathrm{c.m.}} \propto T$. Setting $P_c = P_i$, for $v_g/v_0 \ll 1$ or a sawtooth waveform, the granular temperature scales as $T \propto v_0^2/N_\ell^2$. For symmetric waveforms with $v_g/v_0 \gg 1$ the scaling is $T \propto v_0^2/N_\ell$ [3]. The latter corresponds well to experiments and simulations for one-dimensional systems [12], but not for two-dimensional systems [2,13] or the three-dimensional experiment reported here.

Huntley [4] has suggested that correlations between grain motions at high density reduce the collision frequency below the kinetic-theory result for an ideal gas. The power lost due to collisions is found to scale as $P_c \propto N_\ell^{3/2} v_g^2$. Combining this with the $v_g/v_0 \gg 1$ result for P_i, Huntley found $T \propto v_0^{4/3}/N_\ell^{1/3}$. This is much closer to the experimental and simulation results for two-dimensional systems than is the unmodified kinetic-theory prediction. If Huntley's expression for P_c is set equal to P_i for the $v_g/v_0 \ll 1$ condition appropriate to the present experiments, we obtain $T \propto v_0/N_\ell^{1/2}$. This agrees well with the scaling seen in our experiments. As v_g/v_0 is determined in part by the inelasticity of collisions, this suggests that the degree of inelasticity may be the controlling factor for scaling exponents rather than the system dimension.

Recently, McNamara and Luding [1] carried out a series of two-dimensional simulations in which P_i, P_c, T, and $\Delta h_{\mathrm{c.m.}}$ were measured as the relevant dimensionless parameters were varied. As expected, they found $\Delta h_{\mathrm{c.m.}} \propto T \propto v_0^2$ in the limit of very low density that occurs at high drive velocity v_0. For lower v_0, they found a crossover to $T \propto v_0^{3/2}$ which they traced to a reduction of P_c below its kinetic-theory value when $\Delta h_{\mathrm{c.m.}}$ is small. The detailed mechanism for this reduction is unknown, so it is not clear how these results can be applied to the present three-dimensional experiments. McNamara and Luding also found that $\Delta h_{\mathrm{c.m.}} \propto T$ is not well obeyed at high densities.

The trend found in Ref. [1] of reduced power loss to collisions at high density agrees with Huntley's suggestion, although the functional form may be different. As the power input P_i appears reasonably well understood, we also interpret our experiments as indicating reduced power loss to collisions P_c for dense granular systems in three dimensions, as compared with ideal-gas kinetic-theory predictions.

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