Fields over Unsharp Coordinates

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It has been shown that space-time coordinates can exhibit only very few types of short-distance structures, if described by linear operators: they can be continuous, discrete, or "unsharp" in one of two ways. In the literature, various quantum gravity models of space-time at short distances point towards one of these two types of unsharpness. Here, we investigate the properties of fields over such unsharp coordinates. We find that these fields are continuous—but possess only a finite density of degrees of freedom, similar to fields on lattices. As a special case we recover the Shannon sampling theorem of information theory.

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At the heart of every candidate theory of quantum gravity is an attempt to understand the structure of space-time at very short distances. The reason is a simple gedanken experiment: the latest when trying to resolve distances as small as the Planck scale the accompanying energy-momentum fluctuations due to the uncertainty relation should cause curvature fluctuations large enough to significantly disturb the very space-time distance which one attempts to resolve. Speculations about the resulting behavior of space-time at small distances have ranged from the idea that space-time is discrete, to that it is foamlike, to that space-time may be a derived concept with a highly dynamical short-distance structure, as, e.g., string theory would suggest. At least at present, however, there is no experimental access to sufficiently small scales; and therefore, *a priori*, the short-distance structure of space-time could still be any one out of infinitely many possibilities.

In this context, it has recently been pointed out, in [1], that the range of possible short-distance structures can be reduced to only very few basic possibilities, under a certain assumption. The assumption is that the fundamental theory of quantum gravity possesses for each dimension of space-time an operator X^i which is linear and whose expectation values are real. Note that this assumption is weaker than the usual quantum mechanical assumption of self-adjointness. The dynamics of these X^i may be complicated and the X^i may or may not commute. Nevertheless, one can prove on functional analytic grounds that any such operator X^i, considered separately, describes a coordinate which is necessarily either continuous or discrete, or it is unsharp in one of two well-defined ways. All other cases are mixtures of these.

Since continua and lattices are familiar, we will study one of the two types of unsharp short-distance structures. The second type of unsharpness will be dealt with elsewhere. The type of unsharp coordinate which we will here investigate can be characterized by an uncertainty relation [1]: Such a coordinate is described by an operator X^i for which the formal standard deviation $\Delta X^i = \langle (X^i - \langle X^i \rangle)^2 \rangle^{1/2}$ obeys some positive lower bound:

$$\Delta X^i(\phi) \geq \Delta X^i_{\min}(\langle \phi | X^i | \phi \rangle).$$

Here, ϕ is any unit vector on which the operator can act, and the function $\Delta X^i_{\min}(x)$ describes how the lower bound depends on the X^i-expectation value. If this were nonrelativistic quantum mechanics, the interpretation would be that the X^i coordinate is unsharp in the sense that particles cannot be localized to arbitrary precision on the x^i axis and that the lower bound on the position resolution depends in general on the X^i-expectation value: the localizability of the particle is in general a function of where on the x^i axis one tries to localize the particle. The function $\Delta X^i_{\min}(x)$ may in general also take the value zero, but we will focus on the case where it is strictly positive.

This type of unsharp short-distance structure has indeed frequently appeared in quantum gravity and, in particular, in string theory, in fact from arguments which are independent of our symmetric linear operator assumption. For example, several studies (see, e.g., [2]) suggest that the Heisenberg uncertainty relation may effectively pick up Planck scale or string scale correction terms of the following form:

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \beta(\Delta p)^2 + \ldots]. \qquad (1)$$

For β positive, the lowest order correction in Eq. (1) implies that there is a constant lower bound for Δx, namely, $\Delta x_{\min} = \hbar\sqrt{\beta}$. Of course, it is not necessarily surprising if even quite different candidate quantum gravity theories arrive in this way or another at some positive lower bound $\Delta X^i_{\min}(x)$ on the formal uncertainty in coordinates X^i, because, as we mentioned, for real entities which are described by linear operators this is one out of very few possibilities.

Our aim here is to investigate what this general type of unsharp short-distance structure means in field theory: Is it possible to define fields $\phi(x^i, y)$ "over" such an unsharp coordinate X^i? The operator X^i should act simply as $X^i : \phi(x^i, y) \rightarrow x^i \phi(x^i, y)$, while we let y stand collectively for all other coordinates (if commutative) or any

other quantum numbers. The main question is as follows: How do the fields depend on x^i, given that an unsharp coordinate x^i is neither continuous nor discrete? How does one calculate the Hilbert space scalar product of fields—does it involve an integral over x^i, a sum over discrete points on the x^i axis, or something else?

As we will show here, the answer is that fields $\phi(x^i, y)$ over such unsharp coordinates are indeed well-defined: these fields are continuous functions $\phi(x^i, y)$ over a continuous variable x^i. Crucially, however, these fields are automatically ultraviolet cutoff in the sense that they possess only finitely many degrees of freedom per unit length along the x^i coordinate, similar to fields on lattices.

Before we begin describing the details, let us agree to from now on suppress the index i and the other variables y. We should also mention that some of the operators which describe unsharp coordinates of this type can be represented only on fields which possess isospinor indices, but this phenomenon will be discussed elsewhere.

Let us begin with two definitions: By a *discretization* of the x axis we mean a discrete set of real numbers, $\{x_n\}$, where $x_{n+1} > x_n$ and where n runs through all integers. By a *partitioning* of the x axis we mean a smoothly parametrized family of discretizations $\{x_n(\alpha)\}$ which together make up the entire x axis, namely, such that every point on the x axis, i.e., every real number, occurs in exactly one of the discretizations.

Now our claim is that to each unsharp coordinate X, as characterized by a curve $\Delta X_{\min}(x)$, there corresponds a partitioning $\{x_n(\alpha)\}$ of the x axis such that, if a field $\phi(x)$ is known only on one of the partitioning's discretizations, then the field can already be reconstructed everywhere on the x axis. Namely, if for some arbitrary fixed α the amplitudes $\phi(x_n(\alpha))$ are known for all n, then $\phi(x)$ can be recovered for all x through a reconstruction formula of the form

$$\phi(x) = \sum_n G(x, x_n(\alpha))\phi(x_n(\alpha)) . \tag{2}$$

Thus, the knowledge of a field's amplitudes at finitely many points per unit length along the x axis indeed suffices to describe the field entirely. Thereby, the operation of reconstructing a field is interchangeable with the operation of multiplying it by X:

$$x\phi(x) = \sum_n G(x, x_n(\alpha))x_n(\alpha)\phi(x_n(\alpha)) .$$

The scalar product of two fields (as far as the x dependence is concerned) is a sum:

$$\langle\phi_1|\phi_2\rangle = \sum_n \phi_1^*(x_n(\alpha))\phi_2(x_n(\alpha)) .$$

This scalar product formula gives in fact the same result independently of α, i.e., independently of the choice of discretization on which the sum is being calculated.

Similarly, also the X-expectation value and the second moment of fields can be calculated on any one of the discretizations $\{x_n(\alpha)\}$ and the result does not depend on α. Correspondingly, $\Delta X(\phi) = (\langle\phi|X^2|\phi\rangle - \langle\phi|X|\phi\rangle^2)^{1/2}$ is the standard deviation of the fields' discrete samples on any one of the discretizations $\{x_n(\alpha)\}$ of the x axis. We remark that, more generally, if a field is not only in the domain of X but also in the domain of higher powers of X, say X^r, i.e., if the field decays at infinity with the corresponding inverse power, then the higher moments up to $2r$ are finite, and they too are independent of the discretization in which they are calculated:

$$\langle\phi|X^r|\phi\rangle = \sum_n [x_n(\alpha)]^r \phi^*(x_n(\alpha))\phi(x_n(\alpha)) .$$

We now still need to address the question of exactly how the minimum position uncertainty curve $\Delta X_{\min}(x)$ corresponds to a partitioning of the x axis. One expects, of course, that in regions of the x axis where $\Delta X_{\min}(x)$ is small the spacing needs to be tighter and vice versa.

To see the precise relationship, let us first recall the minimum position uncertainty curve for particles which live on a one-dimensional lattice $\{x_n\}$. Clearly, these particles can be localized to absolute precision $\Delta X = 0$ at each of the lattice sites, say x_{n_0}, namely, with the wave function $\phi(x_n) = \delta_{n,n_0}$. If, however, a particle's expectation value lies in between two lattice sites then its standard deviation cannot be lower than some finite value. As is straightforward to verify, the curve $\Delta X_{\min}(x)$ for a one-dimensional lattice consists of half circles which arc from lattice site to lattice site.

The fields over an unsharp coordinate do not live on only one discretization of the x axis, but simultaneously on a whole family of discretizations which together constitute a partitioning of the x axis. Indeed, as will follow from our main result below, in contrast to ordinary fields over a lattice, fields over unsharp coordinates therefore obey an equation of the following form (for arbitrary fixed α):

$$\sum_n f_n(\alpha)\phi(x_n(\alpha)) = 0 . \tag{3}$$

Equation (3) expresses that, on each one of the discretizations, the fields cannot be too peaked: We will find that $f_n(\alpha) \neq 0$ for all n, which implies, for example, that fields $\phi(x_n) = \delta_{n,n_0}$ do not occur. More precisely, Eq. (3) implies that the variable lower bound $\Delta X_{\min}(x)$ is the joint lower bound of all the minimum X-uncertainty curves of the individual discretizations in the partitioning. Namely, if we denote the minimum X-uncertainty curve of the discretization to the parameter α by $\Delta X_{\min}(x, \alpha)$ [composed of half circles which arc from point $x_n(\alpha)$ to point $x_{n+1}(\alpha)$ for all n] then

$$\Delta X_{\min}(x) = \max_\alpha \Delta X_{\min}(x, \alpha) .$$

In this way, every partitioning $\{x_n(\alpha)\}$ of the x axis determines a minimum position uncertainty curve $\Delta X_{\min}(x)$ and vice versa. We can describe partitionings conveniently by how their lattice spacings vary over the x axis. Indeed, for each partitioning there is a unique lattice spacing function $s(x)$ which obeys for all n and α:

$$s([x_{n+1}(\alpha) + x_n(\alpha)]/2) = x_{n+1}(\alpha) - x_n(\alpha).$$

Its inverse, $\sigma(x) := 1/s(x)$, the "density of degrees of freedom" function, of course also describes an unsharp coordinate entirely.

Interestingly, $s(x)$, $\sigma(x)$, and, correspondingly, the minimum position uncertainty curve $\Delta X_{\min}(x)$ cannot vary arbitrarily abruptly. Intuitively, the reason is clear: if a particle can be localized only to very little precision around one point on the x axis, then it is plausible that the particle cannot be localized to very high precision around a closely neighboring point.

In fact, we find that the possible spatial variability of the unsharpness of a coordinate is constrained to the extent that one discretization, say $\{x_n(0)\}$, together with the set of data $\{\frac{d}{d\alpha} x_n(0)\}$, i.e., together with the discretization's derivative with respect to α, already determines an entire partitioning $\{x_n(\alpha)\}$. {Technically, the discrete amplitudes $v[x_n(0)] := (-1)^n[x_n(0) - i]^{-1}[dx_n/d\alpha(0)]^{1/2}$ belong to a field $v(x)$ which can be reconstructed through Eq. (2), thereby yielding $dx_n(\alpha)/d\alpha$ and therefore $\{x_n(\alpha)\}$ for all values of α.}

Any unsharp coordinate can therefore be specified entirely by specifying one of its discretizations $\{x_n(0)\}$ together with its derivative $\{\frac{d}{d\alpha} x_n(0)\}$. Let us abbreviate these data as $x_n := x_n(0)$ and $x'_n := dx_n(\alpha)/d\alpha|_{\alpha=0}$.

We still need to give explicit expressions for the coefficients $f_n(\alpha)$ of Eq. (3) and of course also for the reconstruction kernel G of Eq. (2). Expressed in terms of the data $\{x_n\}$ and $\{x'_n\}$, we obtain (after lengthy calculation)

$$f_n(0) = (-1)^n \sqrt{x'_n}, \tag{4}$$

and

$$G(x, x_n) = (-1)^{z(x,x_n)} \frac{\sqrt{x'_n}}{x - x_n} \left(\sum_m \frac{x'_m}{(x - x_m)^2} \right)^{-1/2}. \tag{5}$$

Here, $(-1)^{z(x,x_n)}$ provides a sign factor such that $G(x, x_n)$ is continuous in x. The sign factor arises naturally in a product representation:

$$G(x, x_n) = \lim_{N\to\infty} \frac{\prod_{|m|<N, m\neq n} (x - x_m)}{\sqrt{\sum_{|r|<N} \frac{x'_r}{x'_n} \prod_{|s|<N, s\neq r} (x - x_s)^2}}.$$

The proof of these results is rather technical. It is contained in a previous version (see [3]) and will be presented in detail in a follow-up paper. Let us sketch only the proof: The self-adjoint operator $X(0)$ with purely discrete spectrum $\{x_n\}$ possesses simple symmetric restrictions X, each with a $U(1)$ family of self-adjoint extensions $X(\alpha)$. It can be shown that their spectra, $\{x_n(\alpha)\}$, yield partitionings of the real line and that the data $\{x'_n\}$ suffice to specify the restriction and, consequently, the partitioning. The main part of the proof then consists in calculating the unitaries which interpolate the eigenbases of the extensions. The matrix elements of those unitaries constitute the reconstruction kernel.

We eventually arrive at one-parameter resolutions of the Hilbert space identity in terms of an overcomplete and continuously parametrized set of normalizable vectors:

$$1 = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \sum_n |x_n(\alpha)\rangle\langle x_n(\alpha)|$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \frac{d\alpha}{dx} |x\rangle\langle x|.$$

Note that coherent states and continuous wavelets (see, e.g., [4]) yield analogous two-parameter resolutions of the identity.

Let us now consider the instructive special case of unsharp coordinates whose minimum position uncertainty curve $\Delta X_{\min}(x)$ is constant. In this case, also the density of degrees of freedom $\sigma(x)$ is constant, $\sigma = (2\Delta X_{\min})^{-1}$, and the corresponding partitioning $\{x_n(\alpha)\}$ of the x axis reads

$$x_n(\alpha) = 2n\Delta X_{\min} + \alpha.$$

This means that $x_n = x_n(0) = 2n\Delta X_{\min}$ and $x'_n = \frac{dx_n}{d\alpha}(0) = 1$. Applying these parameters in Eq. (5) yields the reconstruction kernel. In this special case, we can use the fact that

$$\sum_n \frac{1}{(z - n)^2} = \left(\frac{\pi}{\sin \pi z} \right)^2$$

to obtain a particularly simple expression for the kernel:

$$G(x, x_n) = \operatorname{sinc}\left(\frac{\pi(x - x_n)}{2\Delta X_{\min}} \right).$$

We observe that the kernel, being a sinc function, is the Fourier transform of the function which is 1 in the frequency interval $[-1/4\Delta X_{\min}, +1/4\Delta X_{\min}]$ and which vanishes everywhere else. This means that the set of fields over a coordinate with constant unsharpness $\Delta X_{\min}$ has a particularly simple characterization: It is the set of fields whose frequency range is limited to the interval $[-\omega_{\max}, \omega_{\max}]$, where $\omega_{\max} = 1/4\Delta X_{\min}$. Also, Eq. (3) acquires a simple interpretation: Eq. (4) yields $f_n(0) = (-1)^n$ so that, as is readily verified, Eq. (3) expresses that the fields' Fourier transforms vanish at $\pm\omega_{\max}$; i.e., Eq. (3) is now a boundary condition in Fourier space.

The fact that functions whose frequency range is within the interval $[-\omega_{\max}, \omega_{\max}]$ can be reconstructed everywhere, via the sinc-function kernel $G(x, x_n) = \operatorname{sinc}[2\pi(x - x_n)\omega_{\max}]$, from their values on discrete points $\{x_n\}$ with spacing $1/2\omega_{\max}$, is indeed well-known, namely, as the Shannon sampling theorem. The sampling spacing $x_{n+1} - x_n = 1/2\omega_{\max}$ is called the Nyquist sampling rate. The basic idea of the theorem was actually already known to Borel (1897) and, according to [5], perhaps even to Cauchy (1841).

Shannon is credited for introducing the theorem into information theory in the 1940s (see [6]): Shannon showed that, due to noise and other limitations, in effect only finitely many amplitude levels of electronic signals can

be resolved, say N. Consequently, for any given ensemble of signals, the measurement of a signal's amplitude at some fixed time t can yield at most $\log_2 N$ bits of information. Crucially now, Shannon's ansatz is to idealize electronic signals $\phi(t)$ as *bandlimited*, i.e., as frequency-limited functions. The sampling theorem then shows that $2\omega_{\max}$ amplitude measurements per unit time suffice to capture such signals entirely—and this implies that these signals can carry information at most at the rate $b = 2\omega_{\max} \log_2 N$ in bits/s or, in terms of the density of degrees of freedom, $b = \sigma \log_2 N$.

The ability provided by the sampling theorem to reconstruct continuous signals from discrete samples and the analysis of their information content have indeed proven very useful in ubiquitous applications from scientific data taking and data analysis to digital audio and video engineering. This, of course, motivated several generalizations of the sampling theorem (see, e.g., [7]). For example, there are methods to improve the convergence of the reconstruction through oversampling (see, e.g., [5]).

One may ask, therefore, why it should have been difficult to generalize the theorem for time-varying information densities. The main reason is that what would seem to be the obvious approach, namely, to try to use Fourier theory to define a notion of time-varying bandwidth, $\omega_{\max}(t)$, faces major difficulties: First, the resolution of a signal's frequency content in time is, of course, limited by the time-frequency uncertainty relation. Second, even low bandwidth signals can actually oscillate arbitrarily fast in any interval of finite size [on these so-called superoscillations (see e.g. [8])].

We avoid those problems by not even trying to define variable bandwidths $\omega_{\max}(t)$ in any Fourier sense. Instead, we obtain a handle on variable information densities through variable densities of degrees of freedom $\sigma(t)$, which are well-defined directly in the time domain. Possible practical applications are currently being explored.

We note that, as a by-product of considering the special case of constant density of degrees of freedom, we have found that the unsharpness of space-time according to the quantum gravity and string theory motivated uncertainty relation, Eq. (1), is indeed of the same type as the unsharpness in the time resolution of bandlimited electronic signals. In fact, it is also the same type as the fundamental unsharpness of optical images since, as is well-known, the aperture induces a bandlimit on the measurement of angles. Of course, to find this type of unsharpness in such different contexts is again not necessarily surprising, given that unsharp real entities described by linear operators—within any arbitrary theory—can exhibit only two types of unsharpness.

Our finding that fields over unsharp coordinates possess finite densities of degrees of freedom can serve, as we saw, as the starting point for an information theoretic analysis of ensembles of fields. This should be interesting to pursue. Indeed, in studies in quantum gravity, and in particular in string theory, the counting of degrees of freedom and an information theoretical perspective have recently found renewed interest, in particular in the contexts of the black hole information loss problem and the holographic principle (see, e.g., [9]).

Our observation that fields over unsharp coordinates are continuous but behave in many ways similar to fields over lattices also raises questions such as how anomalies manifest themselves with this type of ultraviolet cutoff: perhaps through fermion doubling as on lattices, or else? Eventually, it should be possible to work out model independent phenomenological signatures of this type of unsharp space-time. These might be testable if, as recent models of large extra dimension suggest possible, the onset of strong gravity effects is not too far above the currently experimentally accessible scale of about 10^{-18} m, rather than at the Planck scale of 10^{-35} m (see, e.g., [10]).

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