Orbital and Intrinsic Angular Momentum of Single Photons and Entangled Pairs of Photons Generated by Parametric Down-Conversion

H. H. Arnaut[1] and G. A. Barbosa[2,3,*]

[1]*Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte MG, Brazil*

[2]*Laboratório de Óptica Quântica e Não-Linear, Instituto de Física da Universidade de São Paulo, São Paulo SP, Brazil*

[3]*ECE Department, Northwestern University, 2145 N. Sheridan Road, Evanston, Illinois 60208-3118*

(Received 29 November 1999)

States of light that are simultaneously eigenstates of orbital, intrinsic, and total angular momentum are found and eigenstates describing pairs of photons from spontaneous parametric down-conversion reveal classes of photon conversion with angular momentum conservation and cases where the initial angular momentum from the pump laser beam is shared between the converted photons and the generating medium. These angular momentum conservation laws open up a wide range of applications to be explored.

PACS numbers: 42.50.Dv, 42.65.Ky

Since the pioneering experimental work of Beth [1] demonstrating that a beam of circularly polarized light carries angular momentum, other experiments have followed including a report of conservation of photon angular momentum inside a dielectric medium using microwaves [2], transfer of orbital angular momentum to trapped particles [3], and transfer of orbital angular momentum from the pump to the second harmonic generated beam [4]. A description of photons with intrinsic and orbital angular momentum is not obvious or easily demonstrated, neither theoretically nor experimentally. A recent experiment [5] concludes that orbital angular momentum is not conserved in the process of *spontaneous parametric down-conversion* (SPDC) using a Laguerre-Gaussian (LG) pump beam; that is to say, the orbital angular momentum is not transferred to the generated pair of photons.

In this work eigenstates of light with orbital, intrinsic, and total momentum are found and, furthermore, the process of SPDC is revisited and eigenstates of entangled photon pairs (see Klyshko in Ref. [6]) are described, showing how angular momentum *is* conserved between converted photons and the nonlinear crystal.

Part of the interest in twin photons generated by SPDC is that they are a standard source in quantum optics for Einstein-Podolski-Rosen entangled pairs [6,7] and a broad span of applications has been proposed and demonstrated with this nonclassical light including teleportation [8], metrology [9], and quantum images [10,11]. Potential applications include creation of entangled angular momentum states of Bose-Einstein condensates. The possibility of controlling a new spatial degree of freedom of light besides polarization entanglement of photon pairs should not be undervalued and, particularly, orbital angular momentum is a property connected with the less explored *transverse* coherence [10] property of entangled photon pairs.

This Letter presents the different classes of angular momentum conservation using an LG mode pump beam for SPDC and discusses the recent experimental result [5] of nonconservation of orbital angular momentum. Furthermore, it stresses conditions either for complete angular momentum conservation from laser beam photons to entangled photon pairs or exchange of angular momentum between photons and the nonabsorbing nonlinear medium. This exchange process is interpreted as a Raman-like "angular momentum scattering" mediated by spin-flip of an electron pair.

The electromagnetic field is decomposed in continuous modes to avoid problems related to isotropy breaking existing in rectangular quantization volumes [12] and this way the angular momentum operator $\hat{\mathbf{J}}$ is written into the intrinsic part $\hat{\mathbf{S}}$, usually called photon *spin* and the orbital $\hat{\mathbf{L}}$ (see Sec. 10.6.4 of Ref. [12] for definitions):

$$\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}, \quad \text{with } \hat{\mathbf{S}} = \hbar \sum_{s=\pm 1} s \int d^3k \, \frac{\mathbf{k}}{k}\, \hat{n}(\mathbf{k}, s)$$
$$\text{and} \quad \hat{\mathbf{L}} = \frac{-i\hbar}{2} \sum_{s,s'} \int d^3k \{\hat{a}^\dagger(\mathbf{k}, s)\varepsilon_i^*(\mathbf{k}, s) e^{i\omega_{\mathbf{k}} t}[\mathbf{k} \times \nabla_{\mathbf{k}}]\hat{a}(\mathbf{k}, s')\varepsilon_i(\mathbf{k}, s') e^{-i\omega_{\mathbf{k}} t} - \text{H.c.}\}, \tag{1}$$

where $s = 1$ ($s = -1$) represents right-handed (left-handed) polarization and $\mathbf{k}$ is the wave vector.

The angular momentum decomposition (1) is supported by the fact that both components are gauge independent [13] and the intrinsic part is independent of an origin. However, the commutation relationships

$$[\hat{S}_i, \hat{S}_j] = 0, \qquad [\hat{L}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k,$$
$$\text{and} \quad [\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}(\hat{L}_k + \hat{S}_k) \tag{2}$$

show [13] that neither component is a true angular momentum.

 0031-9007/00/85(2)/286(4)$15.00

A search for eigenvectors and eigenvalues of the free electromagnetic field begins with a quite general state for a one photon field

$$|\psi(t)\rangle = \sum_s \int d^3k\, f(\mathbf{k}, s; t)\hat{a}^\dagger(\mathbf{k}, s)\,|0\rangle, \tag{3}$$

followed by application of the angular momentum $\hat{\mathbf{J}}$ given by Eqs. (1) to (3) searching for conditions leading to $\hat{J}_z|\psi(t)\rangle = \alpha|\psi(t)\rangle$. This eigenvalue equation leads to the condition

$$[\mathbf{k} \times \nabla_\mathbf{k} f(\mathbf{k}, s; t)]_z = \frac{i}{\hbar}\,\alpha f(\mathbf{k}, s; t). \tag{4}$$

Equation (4) can be easily solved in cylindrical coordinates, giving $f(\mathbf{k}, s; t) = g(k_\rho, k_z, s; t)\exp(il\phi)$, with $\alpha = l\hbar$. The eigenvalues of $\hat{J}_z$ are then integers and the eigenvectors are given by

$$|\psi(t)\rangle = \sum_s \int d^3k\, g(k_\rho, k_z, s; t)e^{il\phi}\hat{a}^\dagger(\mathbf{k}, s)\,|0\rangle \tag{5}$$

with an associated generic eigenvalue l. The basic condition that $g(k_\rho, k_z, s; t)$ cannot depend on the azimuthal angle ϕ shows that the eigenvectors of $\hat{J}_z$ are associated with a uniform probability distribution around the z axis.

Calculation of the simultaneous eigenvalues of $\hat{S}_z$ and $\hat{L}_z$ (and therefore of $\hat{J}_z$) for a single photon field propagating along the polar angle θ_0 can be found with an eigenfunction described by Eq. (5):

$$|\psi(s = \pm1, \theta_0)\rangle = \int d^3k\, g(k, \theta)\delta(\theta - \theta_0) \times e^{il\phi}\hat{a}^\dagger(\mathbf{k}, s = \pm1)\,|0\rangle. \tag{6}$$

Noninteger eigenvalues are found for $\hat{S}_z$ and $\hat{L}_z$ in a continuous spectrum depending on θ_0 while eigenvalues of $\hat{J}_z$ are integers:

$$\begin{aligned} \hat{S}_z|\psi(s = \pm1, \theta_0)\rangle &= s\cos\theta_0\hbar|\psi(s = \pm1, \theta_0)\rangle, \\ \hat{L}_z|\psi(s = \pm1, \theta_0)\rangle &= (l - s\cos\theta_0)\hbar|\psi(s = \pm1, \theta_0)\rangle, \end{aligned} \tag{7}$$

$$\text{and}\quad \hat{J}_z|\psi(s = \pm1, \theta_0)\rangle = l\hbar|\psi(s = \pm1, \theta_0)\rangle. \tag{8}$$

The form of eigenfunction given in Eq. (5) suggests that the light state describing entangled photon pairs from SPDC may be written in a similar way:

$$|\psi(t)\rangle = \sum_{s,s'} \int_\Omega d^3k \int_{\Omega'} d^3k'\, g(k_\rho, k_z, k'_\rho, k'_z, s, s'; t) \times e^{in\phi}e^{im\phi'}\hat{a}^\dagger(\mathbf{k}, \mathbf{s})\hat{a}^\dagger(\mathbf{k}', s')\,|0\rangle, \tag{9}$$

where $\Omega \cap \Omega' = \varnothing$ and the associated eigenvalue is $n + m$. This possibility will be analyzed starting from the standard interaction Hamiltonian $\hat{H}_I$ for SPDC [6] written under the assumption that the down-converted modes are initially empty and the pump beam is classical:

$$\hat{H}_I = \sum_{s,s'} \int d^3k'\, d^3k\, l_E^{(*)}(\omega_\mathbf{k})l_E^{(*)}(\omega'_\mathbf{k})\hat{a}^\dagger(\mathbf{k}, s)\hat{a}^\dagger(\mathbf{k}', s') \times e^{i(\omega_\mathbf{k} + \omega'_\mathbf{k})t}\chi_{ijk}(\mathbf{e}_{\mathbf{k},s})^*_j(\mathbf{e}_{\mathbf{k}',s'})^*_k \times \int_{V_I} d^3r\, \mathbf{E}_i(\mathbf{r}, t)e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}} + \text{H.c.}, \tag{10}$$

where V_I is the nonlinear interaction volume, $l_E^{(*)}(\omega_\mathbf{k}) = -i\sqrt{\hbar\omega_\mathbf{k}/2\epsilon(\mathbf{k}, s)}$, and $\mathbf{E}(\mathbf{r}, t)$ is the analytical signal associated with the pump beam, which for a Laguerre-Gaussian beam propagating along $\hat{z}$ with the principal component polarized along direction $\hat{x}$ $(= \mathbf{e}_1)$ is given in cylindrical coordinates by

$$\mathbf{E}(\rho, \phi, z; t) \equiv \psi_{lp}(\mathbf{r})e^{i(k_P z - \omega_P t)}\mathbf{e}_1 = \frac{A_{lp}}{\sqrt{1 + (z^2/z_R^2)}}\left[\frac{\sqrt{2}\,\rho}{w(z)}\right]^l L_p^l\left[\frac{2\rho^2}{w^2(z)}\right]\exp\left[i\,\frac{k_P\rho^2}{2q(z)}\right] \times e^{il\arctan(y/x)}\exp\left[-i(2p + l + 1)\arctan\frac{z}{z_R}\right]e^{i(k_P z - \omega_P t)}\mathbf{e}_1, \tag{11}$$

where z_R is the Rayleigh length, $w(z) = w_0\sqrt{1 + z^2/z_R^2}$ (w_0 is the beam radius at the waist $z = 0$), $q(z) = z - iz_R$, and $\rho = \sqrt{x^2 + y^2}$.

The first order approximation for the state vector of the down-converted light reads

$$|\psi(t)\rangle = |0\rangle + \sum_{s,s'} \int d^3k' \int d^3k\, A_{\mathbf{k},s;\mathbf{k}',s'} \times l_E^{(*)}(\omega_\mathbf{k})l_E^{(*)}(\omega'_\mathbf{k})T(\Delta\omega)\tilde{\psi}_{lp}(\Delta\mathbf{k}) \times \hat{a}^\dagger(\mathbf{k}, s)\hat{a}^\dagger(\mathbf{k}', s')\,|0\rangle, \tag{12}$$

where

$$A_{\mathbf{k},s;\mathbf{k}',s'} = A_{lp}\chi_{1jk}[(\mathbf{e}_{\mathbf{k},s})^*_j(\mathbf{e}_{\mathbf{k}',s'})^*_k + (\mathbf{e}_{\mathbf{k}',s'})^*_j(\mathbf{e}_{\mathbf{k},s})^*_k],$$

$T(\Delta\omega) = \exp[i\Delta\omega(t - t_{\text{int}}/2)]\sin(\Delta\omega t_{\text{int}}/2)/(\Delta\omega/2)$ is the time window function defining the $\Delta\omega$ range given the interaction time t_{int}, $\Delta\omega = \omega_\mathbf{k} + \omega'_\mathbf{k} - \omega_P$, $\Delta\mathbf{k} = \mathbf{k} + \mathbf{k}' - \mathbf{k}_P$, and $\tilde{\psi}_{lp}(\Delta\mathbf{k}) = \int_{V_I} d^3r\, \psi_{lp}(\mathbf{r}) \times \exp(-i\Delta\mathbf{k}\cdot\mathbf{r})$.

Assuming the usual conditions that the nonlinear medium is centered on the z axis, the average radius of the beam is small compared to the transverse section of the nonlinear medium and that the crystal length is small compared to the Rayleigh parameter z_R of the beam, one has

$$\tilde{\psi}_{lp}(\Delta\mathbf{k}) = B_{lp}W(\Delta k_z)\varphi_{lp}(k_\rho, k'_\rho, \phi - \phi')\sum_{n=0}^{l}\binom{l}{n} \times (k_\rho)^n(k'_\rho)^{l-n}e^{in\phi}e^{i(l-n)\phi'}, \tag{13}$$

where $B_{lp} = (z_R\pi/k_P^{l+1})[z_R\sqrt{2}/w_0]^l\exp[-\frac{\pi}{2}i(1 - l - p)]2^{p-l+1}$, z_0 is the position of the center of the nonlinear medium, $W(\Delta k_z) = \exp(-i\Delta k_z z_0) \times [\sin(\Delta k_z\ell/2)/(\Delta k_z/2)]$, ℓ is the crystal length, k_ρ and k'_ρ are transverse components of the wave vectors, and

$$\varphi_{lp}(k_\rho, k'_\rho, \phi - \phi') = L_p^l\left[\frac{z_R}{k_P}\rho_k^2\right]\exp\left[-\frac{z_R}{2k_P}\rho_k^2\right] \times \exp\left[-i\frac{z_0}{2k_P}\rho_k^2\right]. \quad (14)$$

The dependence of $\varphi_{lp}(k_\rho, k'_\rho, \phi - \phi')$ on the difference $\phi - \phi'$ can be easily verified by expansion of $\rho_k \equiv |\Delta k_x + i\Delta k_y|$. This function can be developed in the Fourier series

$$\varphi_{lp}(k_\rho, k'_\rho, \phi - \phi') = \sum_{m=-\infty}^{\infty} G_m^{(lp)}(k_\rho, k'_\rho)e^{-im(\phi-\phi')}. \quad (15)$$

Energy and momentum conservation in SPDC are defined by Eq. (12). The longitudinal momentum conservation is specified by the sinc function in (13) while the transverse momentum conservation is defined by the loci of maxima given by all terms containing ρ_k. The electric field, Eq. (11), and all quantities in Eq. (12), are written in the laboratory axis with the propagation direction as the "z" axis. The second order susceptibility tensor χ_{ijk} given in the medium principal axis is assumed to be properly rotated to the lab axis ($\chi_{ijk}^{\text{lab}} = \frac{\partial x'^\alpha}{\partial x^i}\frac{\partial x'^\beta}{\partial x^j}\frac{\partial x'^\gamma}{\partial x^k}\chi_{\alpha\beta\gamma}^{\text{crystal}}$).

The state vector (12) can then be written in the final form resembling Eq. (9):

$$|\psi(t)\rangle = |0\rangle + B_{lp}\sum_{s,s'}\int_\Omega d^3k\int_{\Omega'} d^3k' A_{\mathbf{k},s;\mathbf{k}',s'} l_E^{(*)}(\omega_\mathbf{k}) l_E^{(*)}(\omega'_\mathbf{k}) T(\Delta\omega) W(\Delta k_z) \times \sum_{m=-\infty}^{\infty} G_m^{(lp)}(k_\rho, k'_\rho)\sum_{n=0}^{l}\binom{l}{n}(k_\rho)^n(k'_\rho)^{l-n}e^{i(n-m)\phi}e^{i(l+m-n)\phi'}\hat{a}^\dagger(\mathbf{k},s)\hat{a}^\dagger(\mathbf{k}',s')|0\rangle, \quad (16)$$

and it is assumed in Eq. (16) that the generated photons are in distinct signal and idler modes.

An important question is if Eq. (16) can be cast into the form (9) of eigenvectors of $\hat{J}_z$, that is to say, it will be an eigenvalue of $\hat{J}_z$ only if azimuthal angles are in phase terms in the integrand [see Eqs. (5) and (9)]. In order to simplify the angular dependence of the refractive indexes it is assumed SPDC of type I; however, the same results apply whenever the dependence on azimuthal angles can be neglected either in the photon generation ($A_{\mathbf{k},s;\mathbf{k}',s'}$ term) or in the propagation process (refractive indexes). This is the case for generation whenever $A_{\mathbf{k},s;\mathbf{k}',s'}$ do not depend on the azimuthal angles ϕ and ϕ'. This term encloses the basic process in SPDC and is not related to a particular laser beam shape. This independence of azimuthal angles occurs for specific crystal orientations in the *uniaxial* crystal classes of symmetry C_6, C_4, C_{6v}, and C_{4v} [14]. In these cases of "weak azimuthal dependence" the allowed dependence of the state of the down-converted light on ϕ and ϕ' is already explicit in Eq. (16) and thus

$$|\psi(t)\rangle = |0\rangle + \sum_{m=-\infty}^{\infty}\sum_{n=0}^{l}|J_z = n - m\rangle_{\text{signal}}|J_z = l + m - n\rangle_{\text{idler}} \equiv |0\rangle + \sum_{m=-\infty}^{\infty}\sum_{n=0}^{l} B_{lp}\sum_{s,s'}\int_\Omega d^3k\int_{\Omega'} d^3k' A_{\mathbf{k},s;\mathbf{k}',s'} \times l_E^{(*)}(\omega_\mathbf{k}) l_E^{(*)}(\omega'_\mathbf{k}) T(\Delta\omega) W(\Delta k_z)\binom{l}{n}(k_\rho)^n(k'_\rho)^{l-n} G_m^{(lp)}(k_\rho, k'_\rho)e^{i(n-m)\phi}e^{i(l+m-n)\phi'}\hat{a}^\dagger(\mathbf{k},s)\hat{a}^\dagger(\mathbf{k}',s')|0\rangle. \quad (17)$$

Equation (17) is one of the main results of this work. It shows a nonfactorizable state due to the n and m dependence. In all cases where $A_{\mathbf{k},s;\mathbf{k}',s'}$ do not depend on the azimuthal angles the angular momentum associated to each down-converted pair is then demonstrated to be $l\hbar\mathbf{e}_3$. In other words $\langle\hat{\mathbf{J}}_z\rangle = l\hbar\mathbf{e}_3$ and $\langle(\Delta\hat{\mathbf{J}}_z)^2\rangle = 0$. This null variance expresses perfect angular momentum conservation about the z axis for the photon pair. Besides this, the angular momentum l is not defined for each beam individually but is a property of the photon *pair* as a whole.

Instances where $A_{\mathbf{k},s;\mathbf{k}',s'}$ depends on the azimuthal angles represent a wide number of cases and include biaxial as well as uniaxial crystals (of different symmetries than the ones of weak azimuthal dependence). The biaxial crystal LBO, lithium triborate, utilized in Ref. [5] could not show orbital angular momentum conservation in SPDC because it lacks the necessary symmetry. The symmetry of χ_{ijk}, written in the laboratory axis, will determine $A_{\mathbf{k},s;\mathbf{k}',s'}$. The dependence of $A_{\mathbf{k},s;\mathbf{k}',s'}$ on ϕ and ϕ' can be as simple as products of sin and cos of ϕ and ϕ' *or* can include square roots of trigonometric functions of ϕ and ϕ'. In the simple case $A_{\mathbf{k},s;\mathbf{k}',s'} \Rightarrow A_{lp}\chi_{1jk}(A_1e^{i\phi} + A_{-1}e^{-i\phi})(A'_1e^{i\phi'} + A'_{-1}e^{-i\phi'})$, where A_q and A'_q indicate terms proportional to $\pm 1/2$ or $\mp i/2$. Collecting these terms together with the pure exponential terms in Eq. (16) containing azimuthal angles one obtains (excluding the vacuum)

$$|\psi\rangle = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{l}[|J_{z,\text{signal+idler}} = 2 + l\rangle + |J_{z,\text{signal+idler}} = -2 + l\rangle + |J_{z,\text{signal+idler}} = l\rangle + |J_{z,\text{signal+idler}} = l\rangle]. \quad (18)$$

Equation (18) shows explicitly that angular momentum from the pump beam is either completely conserved in the process of creation of pair of photons in SPDC or the nonlinear medium gains or loses angular momentum in units of $2\hbar$ per annihilated photon of the pump beam in this considered geometry. This angular momentum exchange process can be tentatively interpreted as "angular momentum Raman-like scattering" in the assumed non-absorbing medium. This very fast scattering process (where $\omega_{\mathbf{k}} + \omega'_{\mathbf{k}} = \omega_P$) is done through electron-light interaction and possibly involving a spin-flip of two unpaired electrons of intrinsic angular momentum $\hbar/2$ to produce changes of $\hbar$ per electron.

Because of the very precise character of the eigenvector given by Eq. (17), it can be used for prediction of experiments and applications utilizing this new degree of freedom in several areas such as quantum computing, teleportation of information, or to create entangled angular momentum states of Bose-Einstein condensates. In cryptography, for example, one could add to the polarization entanglement of type II SPDC the detection of signal and idler photons in transverse patterns produced by angular momentum or "l entanglement." In BEC, two condensates could be obtained from a combined cooling starting with a coherent LG mode pump beam and completed by a separated illumination of each condensate by signal and idler photons with entangled orbital angular momentum.

Summarizing, we have found the general form of an eigenvector describing single photons carrying intrinsic and orbital angular momentum. We have detailed the conditions on which angular momentum of light carried by a pump beam is transferred to entangled photon pairs in the process of SPDC. Furthermore, it was also found that for some classes of experiments involving SPDC there are two possibilities, either a complete conservation of angular momentum from the pump beam to the converted photons or an exchange of angular momentum to or from the nonlinear medium resulting in entangled photon pairs carrying $(l \pm 2)\hbar$ of angular momentum.

G. A. Barbosa thanks Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) for support in São Paulo and Dr. R. J. Horowicz for his hospitality.

*Electronic address: barbosa@ray.ece.nwu.edu

[1] R. A. Beth, Phys. Rev. **48**, 471 (1935); **50**, 115 (1936).
[2] M. Kristensen and J. P. Woerdman, Phys. Rev. Lett. **72**, 2171 (1994).
[3] N. B. Simpson, K. Dholakia, L. Allen, and M. J. Padgett, Opt. Lett. **22**, 52 (1997).
[4] K. Dholakia, N. B. Simpson, M. J. Padgett, and L. Allen, Phys. Rev. A **54**, R3742 (1996); J. Courtial, K. Dholakia, L. Allen, and M. J. Padgett, Phys. Rev. A **56**, 4193 (1997).
[5] J. A. Arlt, K. Dholakia, L. Allen, and M. J. Padgett, Phys. Rev. A **59**, 3950 (1999).
[6] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. **124**, 1646 (1961); D. N. Klyshko, Pis'ma Zh. Éksp. Teor. Fiz. **9**, 69 (1969) [JETP Lett. **9**, 40 (1969)]; D. C. Burham and D. L. Weinberg, Phys. Rev. Lett. **25**, 84 (1970); Z. Y. Ou, L. J. Wang, and L. Mandel, Phys. Rev. A **40**, 1428 (1989).
[7] W. Tittel, J. Brendel, N. Gisin, and H. Zbinden, Phys. Rev. A **59**, 4150 (1999).
[8] C. Bennett, G. Brassard, C. Crépeau, R. Josza, A. Peres, and W. K. Wooters, Phys. Rev. Lett. **70**, 1895 (1993); D. Bouwmeester, J. V. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997).
[9] A. Migdall, Phys. Today **52**, 41 (1999), and references therein.
[10] P. H. S. Ribeiro, S. Pádua, J. C. Machado da Silva, and G. A. Barbosa, Phys. Rev. A **49**, 4176 (1994); T. Grayson and G. A. Barbosa, Phys. Rev. A **49**, 2948 (1994); A. Joobeur, B. E. A. Saleh, and M. C. Teich, Phys. Rev. A **50**, 3349 (1994); G. A. Barbosa, Phys. Rev. A **54**, 4473 (1995); E. J. S. Fonseca, C. H. Monken, S. Pádua, and G. A. Barbosa, Phys. Rev. A **59**, 1608 (1999).
[11] L. A. Lugiato, S. Barnett, A. Gatti, I. Marzoli, G. L. Oppo, and H. Wiedemann, in *Coherence and Quantum Optics*, edited by J. Eberly, L. Mandel, and E. Wolf (Plenum Press, New York, 1996), Vol. VII.
[12] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, New York, 1995).
[13] S. J. Van Enk and G. Nienhuis, J. Mod. Opt. **41**, 963 (1994).
[14] F. Zernike and J. E. Midwinter, *Applied Nonlinear Optics* (Wiley, New York, 1973).