Spectral Statistics of Chaotic Systems with a Pointlike Scatterer

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The statistical properties of a Hamiltonian H_0 perturbed by a localized scatterer are considered. We prove that if H_0 describes a bounded chaotic motion, the universal part of the spectral statistics is not changed by the perturbation. This is done first within the random matrix model. Then it is shown by semiclassical techniques that the result is due to a cancellation between diagonal diffractive and off-diagonal periodic-diffractive contributions. The compensation is a very general phenomenon encoding the semiclassical content of the optical theorem.

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In quantum systems, the chaotic or disordered nature of the classical motion is reflected in the statistical properties of the high lying eigenvalues and eigenvectors. For instance, the spectral statistics of ballistic cavities are universal for energy ranges that are small compared to the inverse time of flight through the system. These universal properties are well described by random matrix theory (RMT) [1,2].

Consider a perturbation imposed to a chaotic system. We are interested in the quantum mechanical effects of a particular class of perturbations that are nonclassical, in the sense that almost all the classical trajectories are insensitive to it. If the unperturbed motion is described by a Hamiltonian H_0 acting in an N-dimensional Hilbert space, we consider Hamiltonians of the form

$$H = H_0 + \lambda N |v\rangle\langle v|, \tag{1}$$

where $|v\rangle$ is a fixed vector. N is included in the perturbation for future convenience. The eigenvalues $\{\omega_i\}$ of H satisfy the equation

$$\sum_k \frac{|v_k|^2}{\omega - \epsilon_k} = \frac{1}{\lambda N}, \tag{2}$$

with $\{\epsilon_k\}$ the eigenvalues of H_0 and $v_k = \langle \varphi_k \mid v\rangle$ the amplitudes of $|v\rangle$ in the eigenbasis of H_0.

Rank-one perturbations like in Eqs. (1) and (2) appear in several contexts. The most common one occurs when a local short-range impurity or point scatterer is added to the system [3]. The physical consequences of such a perturbation were studied for Fermi gases [4,5], in the context of RMT [6] and for ballistic motion of particles in regular [7] and chaotic [8] cavities. Another context is the physics of many body problems, where rank-one separable perturbations were considered as a simplified form of residual interaction between the particles in a mean field approach [9]. It is the simplest model leading to collective excitations of the many body system.

A local perturbation is purely wave mechanical. For a system with f degrees of freedom, it represents a modification of the dynamics in a volume $\propto (2\pi\hbar)^f$ in phase space, which tends to zero in the semiclassical limit. For example, the addition of a point scatterer in a ballistic cavity leaves invariant the classical motion while at the quantum level it induces wave effects such as diffraction. The modifications of the eigenvalues produced by the perturbation are described by Eq. (2). The statistical properties of the perturbed spectrum when the unperturbed system H_0 is a regular integrable rectangular billiard were studied by several authors (see, e.g., Refs. [7,10,11]). It was demonstrated that a short range repulsion between the eigenvalues, different from RMT, is induced by the perturbation, thus considerably modifying the initial Poisson distribution. More recently, Sieber [8] has studied, using semiclassical techniques, the modifications by a point scatterer of the spectral statistics of chaotic systems. He showed that diffractive orbits produce finite contributions which may induce deviations with respect to the random matrix model. Whether this deviation really exists for chaotic systems, or on the contrary if there are other (nondiagonal) semiclassical contributions that cancel the purely diffractive terms is the question we answer here.

We prove by two different approaches, namely, a purely statistical model and a semiclassical calculation, that a local perturbation produces no deviations with respect to RMT. In the first place, assuming that the unperturbed eigenvalues and eigenvector components in Eq. (2) are distributed according to RMT, i.e., their joint probability densities are given by [1,2]

$$P(\{\epsilon_k\}) \propto \prod_{i>j} |\epsilon_i - \epsilon_j|^\beta, \tag{3}$$

and

$$P(\{v_k\}) = \prod_i \left(\frac{\beta N}{2\pi}\right)^{1-\beta/2} \exp(-\beta N |v_i|^2/2), \tag{4}$$

we show that the joint probability density for the perturbed eigenvalues is exactly the same as the distribution of the unperturbed ones,

$$P(\{\omega_k\}) \propto \prod_{i>j} |\omega_i - \omega_j|^\beta. \tag{5}$$

Here, $\beta = 1$ (respectively, 2) for systems with (respectively, without) time-reversal symmetry. In the second

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place, and to complete the analysis, a semiclassical calculation of the spectral form factor is considered. The latter is written as a double sum over all the periodic and diffractive orbits of the system. The diffractive orbits are closed trajectories that hit the scatterer. In Ref. [8] the diagonal contribution of the diffractive orbits was obtained [cf. Eq. (14) below]. We compute the off-diagonal contribution coming from the interference of periodic and diffractive orbits, and find that this contribution exactly cancels the diagonal diffractive term. We thus recover the statistics of RMT. The basic physical ingredient responsible for this cancellation is the unitarity of quantum scattering processes, i.e., conservation of the flux scattered by the impurity. Although our semiclassical result is less general than Eq. (5)—it is valid only for the short-time behavior of a two-point function—it applies to a wide class of diffractive systems whose Hamiltonian cannot always be written in the form (1).

In chaotic and disordered systems the local universal fluctuations of the spectrum are described by the Jacobian (3). We ignore here problems related to the confinement of the eigenvalues, which are of minor importance for our purposes. The first ingredient of the proof of Eq. (5) is the joint distribution function of both the old and new eigenvalues, obtained in Ref. [6],

$$P(\{\epsilon_i\},\{\omega_j\}) \propto \frac{\prod_{i>j}(\epsilon_i - \epsilon_j)(\omega_i - \omega_j)}{\prod_{i,j}|\epsilon_i - \omega_j|^{1-\beta/2}}\, e^{-\rho\sum_i(\omega_i-\epsilon_i)},$$

with $\rho = \beta/2\lambda$. We restrict for simplicity to $\lambda > 0$ ($\lambda < 0$ is treated in the same manner). Equation (2) imposes the restrictions $\epsilon_i \le \omega_i \le \epsilon_{i+1}$ (trapping). The distribution for the perturbed eigenvalues, ω_i, is then defined as

$$P(\{\omega_i\}) = \int_{-\infty}^{\omega_1} d\epsilon_1 \int_{\omega_1}^{\omega_2} d\epsilon_2 \cdots \int_{\omega_{N-1}}^{\omega_N} d\epsilon_N\, P(\{\epsilon_i\},\{\omega_j\})$$
$$\propto e^{-\rho\sum_i\omega_i}\prod_{i>j}(\omega_i - \omega_j)W(\beta,\rho), \qquad (6)$$

with

$$W(\beta,\rho) = \int_{-\infty}^{\omega_1}\frac{e^{\rho\epsilon_1}d\epsilon_1}{F(\epsilon_1)}\cdots\int_{\omega_{N-1}}^{\omega_N}\frac{e^{\rho\epsilon_N}d\epsilon_N}{F(\epsilon_N)} \times \prod_{i>j}(\epsilon_i - \epsilon_j),$$

and $F(\epsilon) = \prod_j|\epsilon - \omega_j|^{1-\beta/2}$. Expressing the last product in W as a Vandermonde determinant, and integrating the latter term by term we arrive at

$$W(\beta,\rho) = \det[I_j^{(i-1)}]_{i,j=1,\ldots,N}, \qquad (7)$$

where $I_j^{(i)} = \partial_\rho^i I_j$ is the ith derivative with respect to ρ of

$$I_j = I_j^{(0)} = \int_{\omega_{j-1}}^{\omega_j}\frac{e^{\rho\epsilon}d\epsilon}{F(\epsilon)}. \qquad (8)$$

For $j = 1$, $\omega_{j-1} = -\infty$.

It is straightforward to check that the I_j's satisfy, for any j, the following differential equation [12]:

$$\left[\prod_i(\partial_\rho - \omega_i) + \frac{\beta}{2\rho}\sum_i\prod_{j(\neq i)}(\partial_\rho - \omega_j)\right]I_j = 0. \qquad (9)$$

This differential equation allows one to write

$$I_j^{(N)} = \sum_{i=0}^{N-1} a_i I_j^{(i)},$$

with some coefficients a_i. $W(\beta,\rho)$ as defined in Eq. (7) is the Wronskian of this equation. It then follows that

$$\partial_\rho W = a_{N-1}W, \qquad (10)$$

with $a_{N-1} = \sum_i\omega_i - \beta N/2\rho$. Integration of Eq. (10) leads to

$$W(\beta,\rho) = \frac{W_0}{\rho^{\beta N/2}}\exp\left(\rho\sum_i\omega_i\right).$$

When this result is replaced in Eq. (6) one gets

$$P(\{\omega_i\}) \propto W_0\prod_{i>j}(\omega_i - \omega_j). \qquad (11)$$

W_0 is an integration factor that does not depend on ρ. We compute it from the asymptotic behavior of $I_j^{(i)}$ when $\rho \to \infty$. In this limit the integral in Eq. (8) may be evaluated explicitly,

$$\lim_{\rho\to\infty} I_j \propto \frac{e^{\rho\omega_j}}{\rho^{\beta/2}\prod_{i\neq j}|\omega_i - \omega_j|^{1-\beta/2}}.$$

To leading order $I_j^{(i)} = \omega_j^i I_j$. Inserting this result in Eq. (7) one gets

$$W_0 \propto \prod_{i>j}\frac{|\omega_i - \omega_j|^\beta}{(\omega_i - \omega_j)}.$$

From this equation and Eq. (11) we recover the random matrix distribution function Eq. (5).

A related problem treated previously considers a chaotic system coupled to the environment through a one-channel antenna [13]. The model is equivalent to Eq. (2) but with imaginary λ. For $\lambda \to \infty$ the imaginary part of the perturbed energies is small and Eq. (5) is obtained. Our method, which takes explicit care of the trapping problem, allows one to prove this result for arbitrary λ.

In real physical systems, agreement with random matrix theory is observed in a limited range. This universal behavior concerns correlations over energy ranges that are small compared to $h/T_{\rm min}$, with $T_{\rm min}$ the typical period of the shortest periodic orbit. The above random matrix calculation establishes that the universal part of the spectrum is not changed by the presence of the scatterer. On the other hand, the nonuniversal behavior of the correlation functions occurring at scales of the order of, or larger than, $h/T_{\rm min}$ are modified by the scattering center, since new diffractive orbits are introduced [14,15].

Let us now turn to a semiclassical treatment of the spectral correlations. These are based on trace formula expansions of the density of states $d(\omega) = \sum_k\delta(\omega - \omega_k)$, written as a sum of smoothed plus oscillatory terms $d = \bar{d} + d^{\mathrm{osc}}$. We characterize the correlations by the spectral form factor defined as

$$K(\tau) = \int_{-\infty}^{\infty} \frac{d\eta}{\bar{d}} \left\langle d^{\mathrm{osc}}\left(E + \frac{\eta}{2}\right) d^{\mathrm{osc}}\left(E - \frac{\eta}{2}\right)\right\rangle \times \exp(2\pi i \eta \tau \bar{d}) . \quad (12)$$

The average indicated by brackets is taken over an energy window containing many quantum levels but whose size is small compared to E. We again consider a fully chaotic system with a pointlike scatterer. In the geometrical theory of diffraction $d^{\mathrm{osc}} = d_p^{\mathrm{osc}} + d_d^{\mathrm{osc}}$, where d_p^{osc} and d_d^{osc} are expressed as interferent sums over periodic and diffractive orbits, respectively [14,15],

$$d_{p,d}^{\mathrm{osc}}(E) = \sum_{p,d} A_{p,d} \exp\left(i \frac{S_{p,d}(E)}{\hbar} - i \frac{\pi}{2} \mu_{p,d}\right), \quad (13)$$

with

$$A_p = \frac{T_p}{2\pi\hbar |\det(M_p - 1)|^{1/2}} ,$$
$$A_d = \frac{T_d \mathcal{D}(\vec{n}, \vec{n}') e^{-i\pi(f+1)/4} |\det N|^{1/2}}{4\pi\hbar k (2\pi\hbar)^{(f-1)/2}} .$$

$S_{p,d}(E)$ is the action of the periodic (respectively, diffractive) orbits, $T_{p,d}$ denotes their period, M_p is the monodromy matrix of the periodic orbit, N is the matrix $N_{ij} = \partial^2 S_d / \partial y_i \partial y_j$ (where $\vec{y}$ are coordinates orthogonal to the diffractive trajectory), and μ are the Maslov indices. $\mathcal{D}(\vec{n}, \vec{n}')$ is the scattering amplitude of the scattering center located at $\vec{x}_0$ with incoming $\vec{n}$ and outgoing $\vec{n}'$ directions, defined in terms of the perturbed (G) and unperturbed (G_0) Green's functions by the relation

$$G(\vec{x}, \vec{x}') = G_0(\vec{x}, \vec{x}') + \frac{\hbar^2}{2m} G_0(\vec{x}, \vec{x}_0) \mathcal{D}(\vec{n}, \vec{n}') G_0(\vec{x}_0, \vec{x}') .$$

Using the properties of the periodic orbits of chaotic systems, the diagonal contribution to d_p^{osc} in Eq. (12) gives the short-time random matrix result $K_p(\tau) = (2/\beta)\tau$ [16]. The one scattering contribution of the diffractive orbits in the same approximation is [8]

$$K_d(\tau) = \frac{\tau^2}{8\beta\pi^2} \left(\frac{k}{2\pi}\right)^{2f-4} \sigma , \quad (14)$$

with k the modulus of the wave vector at the impurity and σ its total cross section,

$$\sigma = \int |\mathcal{D}(\vec{n}, \vec{n}')|^2 d\Omega \, d\Omega' \quad (15)$$

($d\Omega$ is the solid angle element). For simplicity, we restrict the calculations to one scattering event (multiple scattering may be considered likewise).

Our purpose is to compute the off-diagonal cross term coming from the product of d_p^{osc} and d_d^{osc} in Eq. (12). The semiclassical expression for this contribution is

$$K_{pd}(\tau) = \frac{2\pi\hbar}{\bar{d}} \left\langle \sum_{p,d} A_p A_d^* \exp[i(S_p - S_d)/\hbar] \times \delta\left(T - \frac{T_p + T_d}{2}\right) + \text{c.c.} \right\rangle . \quad (16)$$

After energy smoothing, K_{pd} has significant contributions only from orbits with close actions $S_p \approx S_d$ (having therefore approximately the same period). Pairs of orbits satisfying this condition may be constructed by considering the neighborhood of the forward scattering orbits. To each periodic orbit passing nearby the scatterer $\mathcal{O}$ we associate an "almost periodic" diffractive orbit that is similar to the periodic orbit but comes back to $\mathcal{O}$ with a slightly different momentum. In Eq. (16) the double sum now involves all the possible pairs of trajectories constructed this way. Consider a surface of section that includes $\mathcal{O}$ and is perpendicular to the momentum of the periodic orbit when it comes nearby to $\mathcal{O}$. Let coordinates measured from $\mathcal{O}$ and momenta in the plane be denoted by $(\vec{q}, \vec{p})$. Consider all the periodic orbits of period T that cut the section through a differential element $d^{f-1}q d^{f-1}p$ located at a distance $\vec{q}$ from $\mathcal{O}$. The difference of action between these periodic orbits and the diffractive orbits associated to them as mentioned above is

$$S_p - S_d = -(1/2) Q_{ij} q_i q_j , \quad (17)$$

with

$$Q_{ij} = \partial^2 S / \partial q_i \partial q_j + \partial^2 S / \partial q_i' \partial q_j + \partial^2 S / \partial q_i \partial q_j' + \partial^2 S / \partial q_i' \partial q_j' ,$$

and $\vec{q}$ ($\vec{q}\,'$) are initial (respectively, final) coordinates on the surface of section. Moreover, one can show that

$$|\det Q| = |\det(M_p - 1) \det N| \cos^2\theta , \quad (18)$$

where θ is the angle between the normal to the surface of section and the momentum of the diffractive orbit.

By generalizing arguments used in the derivation of the Hannay–Ozorio de Almeida sum rule [17] one can prove the following sum rule:

$$\sum_p \frac{\delta(T - T_p) \chi(\vec{q}_p, \vec{p}_p)}{|\det(M_p - 1)|} = \frac{\int d^{f-1}q \, d^{f-1}p \, \chi(\vec{q}, \vec{p})}{\Sigma} , \quad (19)$$

where $\chi(\vec{q}, \vec{p})$ is a test function defined on the surface of section and $(\vec{q}_p, \vec{p}_p)$ are the coordinates of the points at which the periodic orbit p crosses the surface of section. $\Sigma = \int d^f \mathbf{x} \, d^f \mathbf{p} \, \delta(E - H(\mathbf{x}, \mathbf{p}))$ is the total phase-space volume at energy E. From Eq. (16), using Eqs. (17) and (19), we have

$$K_{pd} = \frac{\bar{d}\tau^2 e^{i\pi(f+1)/4}}{\beta k (2\pi\hbar)^{(f-3)/2} \Sigma} \int \sqrt{|\det(M_p - 1)| |\det N|} \times \mathcal{D}^*(\vec{n}, \vec{n}) e^{-(i/2\hbar) Q_{ij} q_i q_j} d^{f-1}q \, d^{f-1}p + \text{c.c.}$$

Integrating the quadratic form in the exponent, taking into account Eq. (18), using the semiclassical density of states $\bar{d} = \Sigma/(2\pi\hbar)^f$, and the fact that the differential element for the momenta may be written $d^{f-1}p = (\hbar k)^{f-1}\cos\theta d\Omega$, one obtains the final expression:

$$K_{pd}(\tau) = \frac{\tau^2}{2\pi\beta}\left(\frac{k}{2\pi}\right)^{f-2}\int i[\mathcal{D}^*(\vec{n},\vec{n}) - \mathcal{D}(\vec{n},\vec{n})]\,d\Omega\,. \quad (20)$$

This is the result for the cross-term contribution. Note that it depends only on $\mathcal{D}(\vec{n},\vec{n})$; this happens because interferent terms between periodic and diffractive orbits can be large only in the forward direction.

The connection with Eq. (14) is made through a general relation valid for the elastic scattering on a finite range potential. The conservation of the flux scattered by the scattering center imposes a relation between the imaginary part of the scattering amplitude and the scattering cross section. This is the well-known optical theorem [18], which in f dimensions takes the form

$$i[\mathcal{D}^*(\vec{n},\vec{n}) - \mathcal{D}(\vec{n},\vec{n})] = -\frac{1}{4\pi}\left(\frac{k}{2\pi}\right)^{f-2} \times \int |\mathcal{D}(\vec{n},\vec{n}')|^2\,d\Omega'.$$

Combining this relation with Eq. (20) one gets our final result:

$$K_{pd}(\tau) = -K_d(\tau)\,. \quad (21)$$

The interference between periodic and diffractive orbits exactly cancels the diagonal contribution of the diffractive orbits, Eq. (14). We recover from semiclassical methods, at least for a two-point function and short times, the RMT result.

The two basic elements producing the cancellation are the sum rule (19) and the optical theorem. Only the former is characteristic of chaotic systems, the latter being very general. The present semiclassical results may be extended by similar methods to multiple scattering events. In a wider context, it should be mentioned that this is one of the rare cases in which a calculation of off-diagonal contributions (whose role is essential in producing the correct result) is done explicitly for chaotic systems.

We have concentrated on the fluctuation properties of eigenvalues of chaotic systems, and have demonstrated that they are unchanged by a local perturbation. This applies to high lying states, where the statistical hypotheses hold. On the opposite extreme, a local perturbation may lead to important modifications of the properties of the ground state of the system. Take, for example, a negative λ. According to Eq. (2), each perturbed eigenvalue is trapped by two unperturbed ones, except the ground state. The energy of the ground state may diminish arbitrarily with increasing $|\lambda|$ and, as can easily be shown, the associated wave function becomes more and more localized at the impurity. In our considerations we have ignored the presence of this "collective" mode.

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