

Resummation of QED Perturbation Series by Sequence Transformations and the Prediction of Perturbative Coefficients

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We propose a method for the resummation of divergent perturbative expansions in quantum electrodynamics and related field theories. The method is based on a nonlinear sequence transformation and uses as input data only the numerical values of a finite number of perturbative coefficients. The results obtained in this way are for alternating series superior to those obtained using Padé approximants. The nonlinear sequence transformation fulfills an accuracy-through-order relation and can be used to predict perturbative coefficients. In many cases, these predictions are closer to available analytic results than predictions obtained using the Padé method.

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Perturbation theory leads to the expansion of a physical quantity $\mathcal{P}(g)$ in powers of the coupling g ,

$$\mathcal{P}(g) \sim \sum_{n=0}^{\infty} c_n g^n. \quad (1)$$

The natural question arises as to how the power series on the right-hand side is related to the (necessarily finite) quantity on the left. It was pointed out in [1] that perturbation theory is unlikely to converge in any Lagrangian field theory. Generically, the asymptotic behavior of the perturbative coefficients is assumed to be of the form [2]

$$c_n \sim K n^\gamma \frac{n!}{S^n}, \quad n \rightarrow \infty, \quad (2)$$

where K , γ , and S are constants. S is related to the first coefficient of the β function of the underlying theory.

In view of the probable divergence of perturbation expansions in higher order, a number of prescriptions have been proposed both for the resummation of divergent perturbation series and for the prediction of higher-order perturbative coefficients. A very important method is the Borel summation procedure whose application to QED perturbation series is discussed in [3,4]. The Borel method, while being useful for the resummation of divergent series, cannot be used for the prediction of higher-order perturbative coefficients in an obvious way.

In recent years, Padé approximants have become the standard tool to overcome problems with slowly convergent and divergent power series [5]. Padé approximants have also been used for the prediction of unknown perturbative coefficients in quantum field theory [6–8]. The $[l/m]$ Padé approximant to the quantity $\mathcal{P}(g)$ represented by the power series (1) is the ratio of two polynomials $P_l(g)$ and $Q_m(g)$ of degree l and m , respectively,

$$[l/m]_{\mathcal{P}}(g) = \frac{P_l(g)}{Q_m(g)} = \frac{p_0 + p_1 g + \cdots + p_l g^l}{1 + q_1 g + \cdots + q_m g^m}.$$

The polynomials $P_l(g)$ and $Q_m(g)$ are constructed so that the Taylor expansion of the Padé approximation agrees with the original input series Eq. (1) up to terms of order $l + m$ in g ,

$$\mathcal{P}(g) - [l/m]_{\mathcal{P}}(g) = O(g^{l+m+1}), \quad g \rightarrow 0. \quad (3)$$

For the recursive computation of Padé approximants we use Wynn's epsilon algorithm [9], which in the case of the power series (1) produces Padé approximants according to $\epsilon_{2k}^{(n)} = [n + k/k]_{\mathcal{P}}(g)$. Further details can be found in Chap. 4 of [10].

In this Letter, we advocate a different resummation scheme. For an infinite series whose partial sums are $s_n = \sum_{j=0}^n a_j$, the nonlinear (Weniger) sequence transformation with initial element s_0 is defined as [see Eq. (8.4-4) of [10]]

$$\delta_n^{(0)}(\beta, s_0) = \frac{\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(\beta+j)_{n-1}}{(\beta+n)_{n-1}} \frac{s_j}{a_{j+1}}}{\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(\beta+j)_{n-1}}{(\beta+n)_{n-1}} \frac{1}{a_{j+1}}}, \quad (4)$$

where $(a)_m = \Gamma(a + m)/\Gamma(a)$ is a Pochhammer symbol. The shift parameter β is usually chosen as $\beta = 1$, and this choice will be exclusively used here (see also [10]). The power of the δ transformation and related transformations [e.g., the Levin transformation, Eq. (7.3-9) of [10]] is due to the fact that explicit estimates for the truncation error of the series are incorporated into the convergence acceleration or resummation process (see Chap. 8 of [10]). Note that the δ transformation (4) has led to numerically stable and remarkably accurate results [11,12] in the resummation of the perturbative series of the quartic, sextic, and octic anharmonic oscillator whose coefficients display a similar factorial pattern of divergence as the quantum field theoretic coefficients indicated in Eq. (2).

We consider as a model problem the QED effective action in the presence of a constant background magnetic field for which the exact nonperturbative result can be

expressed as a proper-time integral:

$$S_B = -\frac{e^2 B^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left\{ \coth s - \frac{1}{s} - \frac{s}{3} \right\} \exp\left(-\frac{m_e^2}{eB} s\right). \quad (5)$$

Here, B is the magnetic field strength, and e is the elementary charge. The general result for arbitrary E and B field can be found in Eq. (3.49) in [13] and in Eq. (4-123) in [14]. The nonperturbative result for S_B can be expanded in powers of the effective coupling $g_B = e^2 B^2/m_e^4$, which results in the divergent asymptotic series

$$S_B \sim -\frac{2e^2 B^2}{\pi^2} g_B \sum_{n=0}^\infty c_n g_B^n, \quad g_B \rightarrow 0. \quad (6)$$

The expansion coefficients

$$[0/0], [1/0], [1/1], \dots, [\nu/\nu], [\nu + 1/\nu], [\nu + 1/\nu + 1], \dots$$

of Padé approximants, which were computed using Wynn's epsilon algorithm [9]. With the help of the notation $\llbracket x \rrbracket$ for the integral part of x , the elements of this sequence of Padé approximants can be written compactly as $\llbracket \llbracket (n+1)/2 \rrbracket / \llbracket n/2 \rrbracket \rrbracket$. Obviously, Padé approximants converge too slowly to the exact result to be numerically useful. The Levin d transformation defined in Eq. (7.3-9) in [10], which is included because it is closely related to the δ transformation (4), fails to accomplish a resummation of the perturbation series, as shown in the fourth column of Table I.

So far, predictions for unknown perturbative coefficients were usually obtained using Padé approximants. The accuracy-through-order relation (3) implies that the Taylor expansion of a Padé approximant reproduces all terms used for its construction. The next coefficient obtained in this way is usually interpreted as the prediction for the first unknown series coefficient (see, e.g., [6-8]).

$$c_n = \frac{(-1)^{n+1} 4^n |\mathcal{B}_{2n+4}|}{(2n+4)(2n+3)(2n+2)}, \quad (7)$$

where \mathcal{B}_{2n+4} is a Bernoulli number, display an alternating sign pattern and grow factorially in absolute magnitude,

$$c_n \sim \frac{(-1)^{n+1}}{8} \frac{\Gamma(2n+2)}{\pi^{2n+4}} [1 + O(2^{-(2n+4)})] \quad (8)$$

as $n \rightarrow \infty$. The series differs from "usual" perturbation series in quantum field theory by the distinctive property that all perturbation theory coefficients are known.

The numerical results in the fifth column of Table I show that the application of the δ transformation (4) to the partial sums $s_n(g_B)$ of the perturbation series (6) produces convergent results even for a coupling constant as large as $g_B = 10$. In the third column of Table I, we display the sequence

The δ transformation (4), when applied to the partial sums $\mathcal{P}_n(g)$ of the power series (1), fulfills the accuracy-through-order relation [11]:

$$\mathcal{P}(g) - \delta_n^{(0)}(1, \mathcal{P}_0(g)) = O(g^{n+2}), \quad g \rightarrow 0. \quad (9)$$

Upon reexpansion of the δ transform a prediction for the next higher-order term in the perturbation series may therefore be obtained.

In Table II we compare predictions for the coefficients c_n of the perturbation series (6) obtained by reexpanding the Padé approximants $\llbracket \llbracket n/2 \rrbracket / \llbracket (n-1)/2 \rrbracket \rrbracket$ and the transforms $\delta_{n-2}^{(0)}(1, s_0(g_B))$, which were computed from the partial sums $s_0(g_B), s_1(g_B), \dots, s_{n-1}(g_B)$. For higher orders of perturbation theory in particular, the Weniger transformation yields clearly the best results, whereas for low orders the improvement over Padé predictions is only gradual. For example, let us assume that for a particular

TABLE I. Resummation of the perturbation series (6) for $g_B = 10$. Results are given in terms of the dimensionless function $\bar{S}_B = 10^2[(8\pi^2)/(-e^2 B^2 g_B)]S_B$. Apparent convergence is indicated by underlining.

n	s_n	$\llbracket \llbracket (n+1)/2 \rrbracket / \llbracket n/2 \rrbracket \rrbracket$	$d_{n-1}^{(0)}(1, s_0(g_B))$	$\delta_{n-1}^{(0)}(1, s_0(g_B))$
1	10.476	10.476 190 476	-2.222 222 222	-2.222 222 222
2	-243.492	-1.617 535 903	-1.617 535 903	-1.617 535 903
3	10 530.918	4.627 654 271	<u>-0.820 833 551</u>	<u>-0.820 833 551</u>
4	-774 888.106	-1.401 288 801	<u>-0.588 575 814</u>	<u>-0.659 817 926</u>
5	8.674 647 $\times 10^7$	2.773 159 300	<u>-0.864 617 071</u>	<u>-0.733 843 307</u>
...
60	-3.652 544 $\times 10^{201}$	<u>-0.920 487 125</u>	5.992 187 $\times 10^{12}$	<u>-0.805 633 981</u>
61	5.553 434 $\times 10^{205}$	<u>-0.400 319 939</u>	1.385 114 $\times 10^{13}$	<u>-0.805 633 980</u>
62	-8.721 566 $\times 10^{209}$	<u>-0.918 054 104</u>	-4.131 495 $\times 10^{13}$	<u>-0.805 633 979</u>
63	1.414 066 $\times 10^{214}$	<u>-0.411 140 364</u>	-8.500 694 $\times 10^{13}$	<u>-0.805 633 978</u>
64	-2.365 759 $\times 10^{218}$	<u>-0.915 746 814</u>	2.890 004 $\times 10^{14}$	<u>-0.805 633 977</u>
65	4.082 125 $\times 10^{222}$	<u>-0.421 331 007</u>	5.272 267 $\times 10^{14}$	<u>-0.805 633 976</u>
66	-7.261 275 $\times 10^{226}$	<u>-0.913 555 178</u>	-2.050 491 $\times 10^{15}$	<u>-0.805 633 975</u>
67	1.330 921 $\times 10^{231}$	<u>-0.430 946 630</u>	-3.296 170 $\times 10^{15}$	<u>-0.805 633 975</u>
...
Exact	-0.805 633 975	-0.805 633 975	-0.805 633 975	-0.805 633 975

TABLE II. Prediction of perturbative coefficients for the power series (6). Results are given for the scaled dimensionless power series $S'_B = [(8\pi^2)/(-e^2 B^2 g_B)]S_B$. First column: order of perturbation theory. Second column: exact coefficients. Third and fourth columns: predictions obtained by reexpanding Padé approximants and Weniger transforms, respectively.

n	Exact	$[[[n/2]]/[(n-1)/2]]$	$\delta_{n-2}^{(0)}(1, s_0(g_B))$
3	+0.107 744 107	+0.050 793 650	+0.050 793 650
4	-0.785 419 025	-0.457 096 214	-0.537 632 214
...
14	$-2.181\,588\,772 \times 10^{15}$	$-2.170\,458\,614 \times 10^{15}$	$-2.181\,574\,607 \times 10^{15}$
15	$+2.055\,682\,756 \times 10^{17}$	$+2.049\,236\,087 \times 10^{17}$	$+2.055\,678\,921 \times 10^{17}$
16	$-2.199\,481\,257 \times 10^{19}$	$-2.194\,962\,521 \times 10^{19}$	$-2.199\,480\,091 \times 10^{19}$
...
24	$-1.711\,360\,421 \times 10^{37}$	$-1.711\,272\,235 \times 10^{37}$	$-1.711\,360\,421 \times 10^{37}$
25	$+4.421\,625\,118 \times 10^{39}$	$+4.421\,484\,513 \times 10^{39}$	$+4.421\,625\,118 \times 10^{39}$
26	$-1.234\,699\,825 \times 10^{42}$	$-1.234\,674\,716 \times 10^{42}$	$-1.234\,699\,825 \times 10^{42}$
...

problem only three coefficients c_0 , c_1 , and c_2 are available and c_3 should be estimated by a rational approximant. Because of the accidental equality $[1/1]_{\mathcal{P}}(g) = \delta_1^{(0)}(1, \mathcal{P}_0(g))$, the predictions for c_3 obtained using the Padé scheme and the δ transformation are equal. Differences between the Padé predictions and those obtained using the δ transformation start to accumulate in higher order.

We now turn to the case of the uniform background electric field, for which the effective action reads [13]

$$S_E = \frac{e^2 E^2}{8\pi^2} \int_0^\infty \frac{ds}{s^2} \left\{ \coth s - \frac{1}{s} - \frac{s}{3} \right\} \times \exp \left[i \left(\frac{m_e^2}{eE} + i\epsilon \right) s \right].$$

This result can be derived from (5) by the replacements $B \rightarrow iE$ and the inclusion of the converging factor. With the convention $g_E = e^2 E^2 / m_e^4$ the divergent asymptotic series

$$S_E \sim \frac{2e^2 E^2}{\pi^2} g_E \sum_{n=0}^\infty c'_n g_E^n, \quad g_E \rightarrow 0, \quad (10)$$

is obtained. The expansion coefficients

$$c'_n = \frac{4^n |\mathcal{B}_{2n+4}|}{(2n+4)(2n+3)(2n+2)} \quad (11)$$

display a nonalternating sign pattern, but are equal in magnitude to the magnetic field case [cf. Eq. (7)]. For physical values of g_E , i.e., for $g_E > 0$, there is a cut in the complex plane, and the nonvanishing imaginary part for S_E gives the pair-production rate. As is well known, resummation procedures for (nonalternating) divergent series usually fail when the coupling g assumes values on the cut in the complex plane [10]. The Borel method fails because of the poles on the integration contour in the Borel integral [4]. The δ transformation and Padé approximations fail for reasons discussed in [10] and [15], respectively.

We now come to an important observation which to the best of our knowledge has not yet been addressed in the literature: the prediction of perturbative coefficients by non-

linear sequence transformations may even work if the resummation of the divergent series fails, i.e., if the coupling g lies on the cut. A general divergent series whose coefficients are nonalternating in sign, evaluated for positive coupling, corresponds to a series with alternating coefficients, evaluated for negative coupling. Alternating series can be resummed with the δ transformation in many cases, and predictions for higher-order coefficients should therefore be possible for both the alternating and the nonalternating case. For example, the perturbative coefficients in Eqs. (7) and (11) differ only in the sign pattern, not in their magnitude. As shown in Table III, rational approximants to the series (6) and (10) produce, after the reexpansion in the coupling, the same predictions up to the different sign pattern.

We stress here that the resummation procedure and the prediction scheme presented in this Letter also work for higher-order terms in the derivative expansion of the QED effective action [16]. The resummation also works for the partition function for the zero-dimensional ϕ^4 theory which is discussed in [14] (p. 464) and is used in [17] as a paradigmatic example for the divergence of perturbative expansions in quantum field theory. Results will be presented in detail elsewhere [16].

TABLE III. Prediction of perturbative coefficients c'_n for the electric background field (10). Results are given for the scaled dimensionless power series $S'_E = [(8\pi^2)/(e^2 E^2 g_E)]S_E$.

n	Exact	$\delta_{n-2}^{(0)}(1, s_0(g_E))$
...
14	$2.181\,588 \times 10^{15}$	$2.181\,574 \times 10^{15}$
15	$2.055\,682 \times 10^{17}$	$2.055\,678 \times 10^{17}$
16	$2.199\,481 \times 10^{19}$	$2.199\,480 \times 10^{19}$
...
24	$1.711\,360 \times 10^{37}$	$1.711\,360 \times 10^{37}$
25	$4.421\,625 \times 10^{39}$	$4.421\,625 \times 10^{39}$
26	$1.234\,699 \times 10^{42}$	$1.234\,699 \times 10^{42}$
...

An interesting and more “realistic” application is given by the β function of the Higgs boson coupling in the standard electroweak model [18]. In the modified minimal subtraction ($\overline{\text{MS}}$) renormalization scheme, five coefficients of this β function are known. Using the first four coefficients, the “prediction” for the fifth coefficient (which is known) may be obtained and compared to the analytic result. Using the transformation $\delta_2^{(0)}$ a prediction of $\beta_4 \approx 4.404 \times 10^7$ is obtained which is closer to the analytic result of $\beta_4 \approx 4.913 \times 10^7$ than the predictions obtained using the $[2/1]$ and $[1/2]$ Padé approximants (these yield $\beta_4 \approx 3.969 \times 10^7$ and $\beta_4 \approx 4.188 \times 10^7$, respectively). The prediction for the unknown coefficient β_5 obtained using $\delta_3^{(0)}$ is $\beta_5 \approx -3.938 \times 10^9$ as compared to $\beta_5 \approx -3.756 \times 10^9$ from the $[2/2]$ Padé approximant.

For the β function of the scalar ϕ^4 theory the situation is similar to the Higgs boson case. Five coefficients are known analytically [19]. Again, the prediction for the fifth coefficient obtained using the transformation $\delta_2^{(0)}$ (1251.3) is closer to the analytic result of 1424.3 than the predictions from the $[2/1]$ and $[1/2]$ Padé approximants which yield values of 1133.5 and 1187.5, respectively. For the unknown sixth coefficient, a prediction of -1.70×10^4 is obtained using $\delta_3^{(0)}$, whereas the $[2/2]$ Padé approximant yields -1.63×10^4 .

We have shown that the δ transformation (4) can be used to accomplish a resummation of alternating divergent perturbation series whose coefficients diverge factorially. In many cases, the δ transforms converge faster to the non-perturbative result than Padé approximants. The δ transformation uses as input data only the numerical values of a finite number of perturbative coefficients. We stress here that the factorial divergence is expected of general perturbative expansions in quantum field theory [see Eq. (2)]. The Weniger δ transformation can be used for the prediction of higher-order coefficients of alternating and non-alternating factorially divergent perturbation series. Both in model problems and in more realistic applications, the δ transformation yields improved predictions (compared to Padé approximants). It appears that the potential of sequence transformations, notably the δ transformation, has not yet been widely noticed in the field of large-order perturbation theory.

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- [1] B. Simon, Phys. Rev. Lett. **28**, 1145 (1972).
- [2] A.I. Vainshtein and V.I. Zakharov, Phys. Rev. Lett. **73**, 1207 (1994).
- [3] V.I. Ogievetsky, Proc. Acad. Sci. USSR **109**, 919 (1956) (in Russian).
- [4] G.V. Dunne and T.M. Hall, Phys. Rev. D **60**, 065002 (1999).
- [5] G.A. Baker and P. Graves-Morris, *Padé Approximants* (Cambridge University Press, Cambridge, 1996), 2nd ed.
- [6] M.A. Samuel, G. Li, and E. Steinfelds, Phys. Rev. D **48**, 869 (1993).
- [7] M.A. Samuel, G. Li, and E. Steinfelds, Phys. Rev. E **51**, 3911 (1995).
- [8] M.A. Samuel, J. Ellis, and M. Karliner, Phys. Rev. Lett. **74**, 4380 (1995); V. Elias, T.G. Steele, F. Chishtie, R. Migneron, and K. Sprague, Phys. Rev. D **58**, 116007 (1998); F. Chishtie, V. Elias, and T.G. Steele, Phys. Lett. B **446**, 267 (1999); F. Chishtie, V. Elias, and T.G. Steele, Phys. Rev. D **59**, 105013 (1999); F. Chishtie, V. Elias, and T.G. Steele, J. Phys. G **26**, 93 (2000).
- [9] P. Wynn, Math. Tables Aids Comput. **10**, 91 (1956).
- [10] E.J. Weniger, Comput. Phys. Rep. **10**, 189 (1989).
- [11] E.J. Weniger, J. Čížek, and F. Vinette, J. Math. Phys. (N.Y.) **34**, 571 (1993).
- [12] E.J. Weniger, Phys. Rev. Lett. **77**, 2859 (1996).
- [13] J. Schwinger, Phys. Rev. **82**, 664 (1951).
- [14] C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
- [15] M. Pindor, Los Alamos e-print hep-th/9903151.
- [16] U.D. Jentschura, J. Becher, M. Meyer-Hermann, P.J. Mohr, E.J. Weniger, and G. Soff (to be published); U.D. Jentschura, E.J. Weniger, and G. Soff, Los Alamos e-print hep-ph/0005198 [J. Phys. G (to be published)]; U.D. Jentschura, Los Alamos e-print hep-ph/0001135 [Phys. Rev. D (to be published)].
- [17] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1996), 3rd ed.
- [18] L. Durand and G. Jaczko, Phys. Rev. D **58**, 113002 (1998).
- [19] H. Kleinert, J. Neu, V. Schulte-Fröhlich, K.G. Chetyrkin, and S.A. Larin, Phys. Lett. B **272**, 39 (1991).