Do Semiclassical Zero Temperature Black Holes Exist?

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The semiclassical Einstein equations are solved to first order in $\epsilon = \hbar/M^2$ for the case of a Reissner-Nordström black hole perturbed by the vacuum stress energy of quantized free fields. Massless and massive fields of spin 0, 1/2, and 1 are considered. We show that in all physically realistic cases, macroscopic zero temperature black hole solutions do not exist. Any static zero temperature semiclassical black hole solutions must then be microscopic and isolated in the space of solutions; they do not join smoothly onto the classical extreme Reissner-Nordström solution as $\epsilon \to 0$.

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Static spherically symmetric zero temperature black holes have proven to be very interesting and important at the classical, semiclassical, and quantum levels. Classically the only static spherically symmetric black hole solution to Einstein's equations with zero surface gravity (and hence zero temperature) is the extreme Reissner-Nordström (ERN) black hole, which possesses a charge equal in magnitude to its mass. At the quantum level, the statistical mechanical entropy of zero temperature (extreme) black holes has been calculated in string theory [1] and shown to be identical to the usual Bekenstein-Hawking formula for the thermodynamic entropy. The usual semiclassical temperature and entropy calculations for ERN black holes have all been made in the test field approximation where the effects of quantized fields on the spacetime geometry are not considered. However, it is well known that quantum effects alter the spacetime geometry near the event horizon of a black hole. In particular, they can change its surface gravity and hence its temperature [2–6].

In this Letter we examine the effects of the semiclassical backreaction due to the vacuum stress energy of massless and massive free quantized fields with spin 0, 1/2, and 1 on a static Reissner-Nordström (RN) black hole. Our focus is on the effects the fields have on macroscopic black holes, those substantially larger than the Planck mass. We are specifically interested in those macroscopic black hole configurations that may have zero temperature when semiclassical effects are incorporated. Such configurations must be nearly extreme; that is, they must have a charge to mass ratio near unity. The fields are assumed to be in the Hartle-Hawking state, which is a thermal state at the black hole temperature. At the event horizon the stress energy of quantized fields in the Hartle-Hawking state should be of order $\epsilon = \hbar/M^2$ compared to the stress energy of the classical electric field, with M the mass of the black hole. Thus semiclassical effects may be handled using perturbation theory.

In all physically realistic cases we find that solutions to the perturbed semiclassical backreaction equations corresponding to static spherically symmetric zero temperature black holes do not exist. In the context of semiclassical gravity with free quantized fields as the matter source, this means that no macroscopic zero temperature static black hole solutions exist. This is a very surprising and general result that may have significant implications for black hole thermodynamics. If there are any zero temperature static black hole solutions within the full semiclassical theory of gravity (not perturbation theory), then those solutions must be isolated in the space of solutions from the classical extreme Reissner-Nordström solution. That is, they cannot join smoothly onto the ERN solution as $\hbar/M^2$ approaches zero.

The general static spherically symmetric metric can be written in the form [7]

$$ds^2 = -f(r)\,dt^2 + h(r)\,dr^2 + r^2\,d\Omega^2, \qquad (1)$$

where $d\Omega^2$ is the metric of the two-sphere. The metric can describe a black hole with an event horizon at $r = r_h$ if $f(r_h) = 0$. To avoid having a scalar curvature singularity at the event horizon it is necessary that $h^{-1}(r_h) = 0$ as well [8]. The surface gravity of such a black hole is

$$\kappa = \left(\frac{1}{2}\right)\frac{f'}{\sqrt{fh}}\bigg|_{r=r_h}, \qquad (2)$$

where the prime represents a derivative with respect to r and the expression is evaluated at the horizon radius, r_h. The temperature is then [9] $T = \kappa/(2\pi)$.

Since we wish to perturb the spacetime with the vacuum energy of quantized fields, we begin by considering the general Reissner-Nordström metric as the "bare" state. For the RN metric,

$$f(r) = h^{-1}(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad (3)$$

where Q is the electric charge and M is the mass of the black hole. The outer event horizon is located at

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$$r_+ = M + \sqrt{M^2 - Q^2}. \tag{4}$$

For the ERN black hole $|Q| = M$.

In semiclassical gravity, the geometry is treated classically while the matter fields are quantized. In examining the semiclassical perturbations of the RN metric caused by the vacuum energy of quantized fields, we continue to treat the background electromagnetic field as a classical field. The right-hand sides of the semiclassical Einstein equations will then contain both classical and quantum stress-energy contributions,

$$G^{\mu}{}_{\nu} = 8\pi[(T^{\mu}{}_{\nu})^C + \langle T^{\mu}{}_{\nu}\rangle]. \tag{5}$$

We consider the situation where the black hole is in thermal equilibrium (whether at zero or nonzero temperature) with the quantized field; the perturbed geometry then continues to be static and spherically symmetric. To first order in $\epsilon = \hbar/M^2$ the general form of the perturbed RN metric may be written as

$$ds^2 = -[1 + 2\epsilon\rho(r)]\left[1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}\right]dt^2 + \left[1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}\right]^{-1} dr^2 + r^2\, d\Omega^2. \tag{6}$$

The function $m(r)$ contains both the classical mass and a first-order quantum perturbation,

$$m(r) = M[1 + \epsilon\mu(r)]. \tag{7}$$

The metric perturbation functions, $\rho(r)$ and $\mu(r)$, are determined by solving the semiclassical Einstein equations expanded to first order in ϵ,

$$\frac{d\mu}{dr} = -\frac{4\pi r^2}{M\epsilon}\langle T^t{}_t\rangle, \tag{8}$$

$$\frac{d\rho}{dr} = \frac{4\pi r}{\epsilon}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}[\langle T^r{}_r\rangle - \langle T^t{}_t\rangle]. \tag{9}$$

The right-hand side of Eq. (9) is divergent on the horizon unless $[\langle T^r{}_r\rangle - \langle T^t{}_t\rangle]$ vanishes there in the RN case and unless it and its first radial derivative vanish there in the ERN case. Conservation of stress energy implies that so long as the radial derivative of the stress-energy tensor is finite at the horizon then there is no divergence in the (nonextreme) RN case. For the extreme case in two dimensions, Trivedi has shown [10] that a divergence of this quantity does occur for the conformally invariant scalar field. However in four dimensions Anderson, Hiscock, and Loranz have shown [11] by explicit numerical computation of the renormalized stress energy of a quantized massless scalar field that there is no divergence in the stress energy at the horizon. If such a divergence did occur in four dimensions for some other field, it would indicate that a freely falling observer passing through the event horizon would see an infinite energy density there. The perturbation approximation would break down in this case, even for ERN black holes with arbitrarily large masses, and hence would be outside of the scope of this work.

Assuming the perturbation expansion remains valid, the functions $\mu(r)$ and $\rho(r)$, obtained by integrating Eqs. (8) and (9), will contain constants of integration. It is convenient to define them as the values of the metric perturbations on the unperturbed horizon at r_+, so that $\mu(r_+) = C_1$ and $\rho(r_+) = C_2$. Since we are working in perturbation theory, the values of these quantities on the actual horizon are, to leading order, also C_1 and C_2, respectively. Then to first order in ϵ the value of $m(r)$ at the horizon is $m(r_h) = M(1 + \epsilon C_1)$. It is clear that C_1 represents a finite renormalization of the mass M of the black hole. As in previous work [5,6], we hereafter denote the renormalized perturbed mass at the horizon, $m(r_h) = M(1 + \epsilon C_1)$, by M_R. The quantity $[1 - 2m(r_h)/r_h + Q^2/r_h^2]$ then vanishes at $r_h = r_+$, where now $r_+ = M_R + (M_R^2 - Q^2)^{1/2}$. Thus, this renormalization causes the perturbed horizon to be located at the same radius r_+ (as a function of the physical, renormalized mass M_R and the charge Q) as the classical horizon.

To decide whether a semiclassically perturbed black hole has zero temperature, we must calculate the surface gravity of the perturbed metric to first order in ϵ. Applying Eq. (2) to the metric of Eq. (6) and using Eqs. (8) and (9) to simplify the result gives

$$\kappa = \frac{\sqrt{M_R^2 - Q^2}}{r_+^2}(1 + \epsilon C_2) + 4\pi r_+ \langle T^t{}_t\rangle|_{r=r_+}. \tag{10}$$

Now consider semiclassical black holes that, at first order in ϵ, have precisely zero temperature. Such black holes are legitimate solutions within the context of perturbation theory only if they maintain zero temperature as ϵ is reduced to zero. From Eq. (10) it is seen that for the surface gravity, κ, to be zero, the classical surface gravity of the bare black hole,

$$\kappa_0 = \frac{\sqrt{M_R^2 - Q^2}}{r_+^2}, \tag{11}$$

must be at most of order ϵ. Thus, the term in Eq. (10) involving the (unknown) integration constant C_2 will be at least of order ϵ^2 and hence may be discarded in this case. The total surface gravity of the semiclassical solution at first order then involves two terms: the classical surface gravity, which is always non-negative, and a term proportional to $\langle T^t{}_t\rangle$. To have a semiclassically perturbed zero temperature black hole, it is then necessary that $\langle T^t{}_t\rangle$ be nonpositive at the horizon. This implies that the vacuum energy density at the event horizon must be non-negative. If the vacuum energy density is negative at the event horizon (and therefore the weak energy condition [9] is violated there), then quantum effects will prevent a zero temperature semiclassical perturbed black hole from existing.

The calculation of the expectation value of the stress energy of a quantized field in a curved spacetime is a very difficult exercise. However, the problem is simplified in the present case by our focus on zero temperature solutions. Since the classical bare solution must have a surface gravity that is of order ϵ or less, we can simply consider the vacuum stress energy for the ERN spacetime. While the actual bare spacetime may be slightly nonextreme (to order ϵ), the differences between the vacuum stress-energy tensor of the extreme spacetime and the bare spacetime will be of order ϵ^2 and may be ignored.

The ERN spacetime is asymptotically congruent to the conformally flat Robinson-Bertotti spacetime as one approaches the event horizon at $r = M_R$ [12–14]. The vacuum stress energy of a quantized field should similarly asymptotically approach the Robinson-Bertotti values as one approaches the event horizon of the ERN spacetime. This has been confirmed numerically for the scalar field using point splitting renormalization [11]. For conformally invariant (hence, massless) quantized fields, the vacuum stress energy in the Robinson-Bertotti spacetime may be obtained using the results of Brown and Cassidy [15] and Bunch [16]. It is [11]

$$\langle T^{\mu}{}_{\nu}\rangle = \frac{b(s)}{2880\pi^2 M^4}\,\delta^{\mu}{}_{\nu}\,, \tag{12}$$

with $b(s) = 1$, $\frac{11}{2}$, and 62 for scalar, spinor, and vector fields, respectively. Since $\langle T^t{}_t\rangle$ is positive for all three of these cases, the vacuum energy density is negative in all these cases on the ERN horizon, and hence there are no zero temperature linearly perturbed RN black holes associated with conformally invariant quantized fields.

Next let us consider the massless quantized scalar field with arbitrary curvature coupling, ξ (the scalar field is conformally invariant only if $\xi = 1/6$). In this case, the vacuum stress-energy tensor has been numerically computed using point splitting renormalization for the ERN black hole spacetime [11]. The vacuum stress energy depends on ξ in a linear fashion and may be divided into conformal and nonconformal pieces:

$$\langle T^{\mu}{}_{\nu}\rangle = C^{\mu}{}_{\nu} + \left(\xi - \frac{1}{6}\right)D^{\mu}{}_{\nu}\,. \tag{13}$$

Anderson, Hiscock, and Loranz [11] found that $C^{\mu}{}_{\nu}$ approaches the Robinson-Bertotti values as $r \to M$ and that all components of $D^{\mu}{}_{\nu}$ approach zero in that limit. Hence, at the horizon of an ERN black hole, the vacuum stress-energy tensor of a quantized scalar field is independent of the curvature coupling and is equal to the Robinson-Bertotti value. Therefore, there are no zero temperature linearly perturbed RN black holes associated with massless quantized scalar fields for any value of the curvature coupling.

We also wish to consider quantized massive fields in the ERN black hole spacetime. The vacuum stress energy of quantized massive fields in the RN spacetime has been numerically computed using point splitting renormalization in the case of scalar fields, by Anderson, Hiscock, and Samuel [17]. They also developed the DeWitt-Schwinger approximation $\langle T^{\mu}{}_{\nu}\rangle_{\rm DS}$ for the stress energy of the massive scalar field and found that the exact values of the stress-energy components were well approximated when the black hole mass M and field mass m satisfy $Mm > 2$ (it does not matter here whether M is the bare or renormalized black hole mass; any resulting difference will be higher order in ϵ). As the field mass is increased, the DeWitt-Schwinger approximation rapidly becomes more accurate. The DeWitt-Schwinger approximate value for the vacuum energy density of a massive scalar field, evaluated at the event horizon of an ERN black hole is

$$\langle T^t{}_t\rangle_{\rm DS}|_{r=M} = \frac{\epsilon(5 - 14\xi)}{10\,080\pi^2 M^4 m^2}\,. \tag{14}$$

Zero temperature perturbed solutions will be possible only if $\langle T^t{}_t\rangle$ is negative. Examination of Eq. (14) shows that will be possible only if $\xi \geq \frac{5}{14}$, a range that excludes the cases of greatest physical interest, namely the minimally ($\xi = 0$) and conformally ($\xi = 1/6$) coupled fields. A thorough study of RN black holes (with arbitrary charge) perturbed by a quantized massive scalar field has been presented elsewhere [6].

The DeWitt-Schwinger approximation has recently been extended to the case of massive spinor and vector fields in the RN black hole spacetime by Matyjasek [18]. The accuracy of the DeWitt-Schwinger approximation is unknown in this case, as no direct calculation of the exact value of $\langle T^{\mu}{}_{\nu}\rangle$ has been performed for these fields in the RN spacetime. For the spinor field around an ERN black hole, Matyjasek finds

$$\langle T^t{}_t\rangle_{\rm DS}|_{r=M} = \frac{37\epsilon}{40\,320\pi^2 M^4 m^2}\,, \tag{15}$$

while for the vector field, he obtains

$$\langle T^t{}_t\rangle_{\rm DS}|_{r=M} = \frac{19\epsilon}{3360\pi^2 M^4 m^2}\,. \tag{16}$$

Since both of these values for $\langle T^t{}_t\rangle$ are manifestly positive, it appears that perturbations of an ERN black hole caused by quantized massive spinor or vector fields cannot yield a zero temperature solution.

Finally we note that in general there are higher derivative terms in the semiclassical backreaction equations which come from terms in the gravitational action that are quadratic in the curvature. These terms can be taken into account perturbatively by putting them on the right-hand side of the equations and evaluating them in the background geometry [19]. The effective stress-energy tensor for these terms vanishes at the event horizon in the ERN geometry. Thus, these terms cannot cancel the effects of the negative energy densities due to the quantized fields.

Our results imply that if static zero temperature semiclassical black hole solutions do exist, they must not smoothly join onto the classical zero temperature ERN solution as $\epsilon = \hbar/M^2 \to 0$. This suggests that any such solutions are truly microscopic, with masses within a few orders of the Planck mass. Whether such small zero temperature black hole solutions exist remains an open question.

One implication of the nonexistence of macroscopic zero temperature black hole solutions is that, for fixed mass M, there is a minimum temperature that any static spherically symmetric semiclassical black hole can have, namely [from Eq. (10)], $T = 2r_+\langle T^t{}_t\rangle|_{r=r_+}$. Thus, it is not only impossible to build a macroscopic zero temperature black hole [20], it is impossible to build one that is arbitrarily close to zero temperature. This is a reformulation of one version of the third law of black hole mechanics [9].

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[1] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
[2] J. M. Bardeen, Phys. Rev. Lett. **46**, 382 (1981).
[3] J. W. York, Jr., Phys. Rev. D **31**, 775 (1985).
[4] D. Hochberg, T. W. Kephart, and J. W. York, Jr., Phys. Rev. D **48**, 479 (1993).
[5] P. R. Anderson, W. A. Hiscock, J. Whitesell, and J. W. York, Jr., Phys. Rev. D **50**, 6427 (1994).
[6] B. E. Taylor, W. A. Hiscock, and P. R. Anderson, Phys. Rev. D. **61**, 84021 (2000).
[7] Throughout we use units such that $\hbar = c = G = k_B = 1$. Our conventions are those of C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[8] P. R. Anderson and C. D. Mull, Phys. Rev. D. **59**, 044007 (1999).
[9] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[10] S. P. Trivedi, Phys. Rev. D **47**, 4233 (1993).
[11] P. R. Anderson, W. A. Hiscock, and D. J. Loranz, Phys. Rev. Lett. **74**, 4365 (1995).
[12] B. Bertotti, Phys. Rev. **116**, 1331 (1959).
[13] I. Robinson, Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys. **7**, 351 (1959).
[14] B. Carter, in *Black Holes,* edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973).
[15] L. S. Brown and J. P. Cassidy, Phys. Rev. D **15**, 2810 (1977).
[16] T. S. Bunch, J. Phys. A **12**, 517 (1979).
[17] P. R. Anderson, W. A. Hiscock, and D. A. Samuel, Phys. Rev. D **51**, 4337 (1995).
[18] J. Matyjasek, Phys. Rev. D **61**, 124019 (2000).
[19] M. Campanelli, C. O. Lousto, and J. Audretsch, Phys. Rev. D **49**, 5188 (1994); Phys. Rev. D **51**, 6810 (1995).
[20] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).