Monopole-Antimonopole Solutions of Einstein-Yang-Mills-Higgs Theory

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We construct static axially symmetric solutions of SU(2) Einstein-Yang-Mills-Higgs theory in the topologically trivial sector, representing gravitating monopole-antimonopole pairs, linked to the Bartnik-McKinnon solutions.

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Introduction.—The SU(2) Yang-Mills-Higgs (YMH) theory possesses monopole [1], multimonopole [2–4], and monopole-antimonopole pair solutions [5–7]. The magnetic charge of these solutions is proportional to their topological charge. While monopole and multimonopole solutions reside in topologically nontrivial sectors, the monopole-antimonopole pair solution is topologically trivial.

When gravity is coupled to the YMH theory, a branch of gravitating monopole solutions emerges smoothly from the monopole solution of flat space [8–10]. The coupling constant α, entering the Einstein-Yang-Mills-Higgs (EYMH) equations, is proportional to the gravitational constant G and to the square of the Higgs vacuum expectation value η. The monopole branch ends at a critical value $\alpha_{\rm cr}$, beyond which gravity becomes too strong for regular monopole solutions to persist, and collapse to charged black holes is observed [8–10]. Indeed, when the critical value $\alpha_{\rm cr}$ is reached, the gravitating monopole solutions develop a degenerate horizon [11], and the exterior spacetime of the solution corresponds to that of an extremal Reissner-Nordstrøm (RN) black hole with unit magnetic charge [8–10,12].

Besides the fundamental gravitating monopole solution, EYMH theory possesses radially excited monopole solutions not present in flat space [8–10]. These excited solutions also develop a degenerate horizon at some critical value of the coupling constant, but they shrink to zero size in the limit $\alpha \to 0$. Rescaling of the solutions reveals that in this limit the Bartnik-McKinnon (BM) solutions [13] of the Einstein-Yang-Mills (EYM) theory are recovered. For the excited solutions the limit $\alpha \to 0$ therefore corresponds to the limit of vanishing Higgs expectation value, $\eta \to 0$.

In this Letter we investigate how gravity affects the static axially symmetric monopole-antimonopole pair (MAP) solution of flat space [6,7], and we elucidate that curved space supports a rich spectrum of MAP solutions not present in flat space.

In particular, we show that, with increasing α, a branch of gravitating MAP solutions emerges smoothly from the flat space MAP solution and ends at a critical value $\alpha_{\rm cr}^{(1)}$, when gravity becomes too strong for regular MAP solutions to persist. But while the branch of monopole solutions can merge into an extremal RN black hole solution at the critical α, there are no Schwarzschild solutions with degenerate horizon into which the MAP solutions could merge. Indeed we find that at $\alpha_{\rm cr}^{(1)}$ a second branch of MAP solutions emerges, extending back to $\alpha = 0$. Along this upper branch the MAP solutions shrink to zero size, in the limit $\alpha \to 0$, and approach the BM solution with one node (after rescaling).

Since the BM solution with one node is related to a branch of MAP solutions, one immediately concludes that the excited BM solutions with k nodes are related to branches of excited MAP solutions. Indeed, constructing the first excited MAP solution by starting from the BM solution with two nodes, we find that it represents a MAP solution, possessing two monopole-antimonopole pairs.

Axially symmetric ansatz.—The static axially symmetric MAP solutions of SU(2) EYMH theory with action

$$S = \int \left[\frac{R}{16\pi G} - \frac{1}{2e^2}\,{\rm Tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}\,{\rm Tr}(D_\mu \Phi D^\mu \Phi) \right] \sqrt{-g}\, d^4x \,, \qquad (1)$$

with Yang-Mills coupling constant e and vanishing Higgs self-coupling, are obtained in isotropic coordinates with metric [14]

$$ds^2 = -f dt^2 + \frac{m}{f}(dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2\theta \, d\varphi^2 , \qquad (2)$$

where f, m, and l are functions of only r and θ. The MAP ansatz reads for the purely magnetic gauge field ($A_0 = 0$) [6,7]

$$A_\mu dx^\mu = \frac{1}{2e} \left\{ \left[\frac{H_1}{r} dr + 2(1 - H_2) d\theta \right] \tau_\varphi - 2 \sin\theta \left[H_3 \tau_r^{(2)} + (1 - H_4)\tau_\theta^{(2)} \right] d\varphi \right\} , \qquad (3)$$

and for the Higgs field

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$$\Phi = (\Phi_1 \tau_r^{(2)} + \Phi_2 \tau_\theta^{(2)}), \tag{4}$$

with su(2) matrices (composed of the standard Pauli matrices τ_i)

$$\begin{aligned} \tau_r^{(2)} &= \sin 2\theta \tau_\rho + \cos 2\theta \tau_3, \\ \tau_\theta^{(2)} &= \cos 2\theta \tau_\rho - \sin 2\theta \tau_3, \\ \tau_\rho &= \cos\varphi \tau_1 + \sin\varphi \tau_2, \\ \tau_\varphi &= -\sin\varphi \tau_1 + \cos\varphi \tau_2. \end{aligned} \tag{5}$$

The four gauge field functions H_i and the two Higgs field functions Φ_i depend only on r and θ. We fix the residual gauge degree of freedom [3,6,7,14] by choosing the gauge condition $r\partial_r H_1 - 2\partial_\theta H_2 = 0$ [7].

To obtain regular asymptotically flat solutions with finite energy density we impose at the origin ($r = 0$) the boundary conditions

$$\begin{aligned} H_1 = H_3 = H_2 - 1 = H_4 - 1 &= 0, \\ \sin 2\theta \Phi_1 + \cos 2\theta \Phi_2 &= 0, \\ \partial_r(\cos 2\theta \Phi_1 - \sin 2\theta \Phi_2) &= 0, \\ \partial_r f = \partial_r m = \partial_r l &= 0. \end{aligned}$$

On the z axis the functions H_1, H_3, Φ_2 and the derivatives $\partial_\theta H_2, \partial_\theta H_4, \partial_\theta \Phi_1, \partial_\theta f, \partial_\theta m, \partial_\theta l$ have to vanish, while on the ρ axis the functions $H_1, 1 - H_4, \Phi_2$ and the derivatives $\partial_\theta H_2, \partial_\theta H_3, \partial_\theta \Phi_1, \partial_\theta f, \partial_\theta m, \partial_\theta l$ have to vanish. For solutions with vanishing net magnetic charge the gauge potential approaches a pure gauge at infinity. The corresponding boundary conditions for the fundamental MAP solution are given by [6,7]

$$\begin{aligned} H_1 = H_2 = 0, \quad H_3 = \sin\theta, \quad 1 - H_4 = \cos\theta, \\ \Phi_1 = \eta, \quad \Phi_2 = 0, \quad f = m = l = 1. \end{aligned} \tag{6}$$

Introducing the dimensionless coordinate $x = r\eta e$ and the Higgs field $\phi = \Phi/\eta$, the equations depend only on the coupling constant α, $\alpha^2 = 4\pi G\eta^2$. The mass M of the MAP solutions can be obtained directly from the total energy-momentum "tensor" $\tau^{\mu\nu}$ of matter and gravitation, $M = \int \tau^{00}\, d^3r$ [15], or equivalently from $M = -\int (2T_0{}^0 - T_\mu{}^\mu)\sqrt{-g}\, dr\, d\theta\, d\varphi$, yielding the dimensionless mass $\mu = \frac{e}{4\pi\eta} M$.

Results.—Subject to the above boundary conditions, we solve the equations numerically [16]. In the limit $\alpha \to 0$, the lower branch of gravitating MAP solutions emerges smoothly from the flat space solution [6,7]. The modulus of the Higgs field of these MAP solutions possesses two zeros, $\pm z_0$, on the z axis, corresponding to the location of the monopole and antimonopole, respectively.

With increasing α the monopole and antimonopole move closer to the origin, and the mass μ of the solutions decreases. The lower branch of MAP solutions ends at the critical value $\alpha_{\rm cr}^{(1)} = 0.670$. In Fig. 1 we show the energy density $\varepsilon = -T_0{}^0 = -L_M$ of the MAP solution at $\alpha_{\rm cr}^{(1)}$. It possesses maxima on the positive and negative z axis close to the locations of the monopole and antimonopole and a saddle point at the origin.

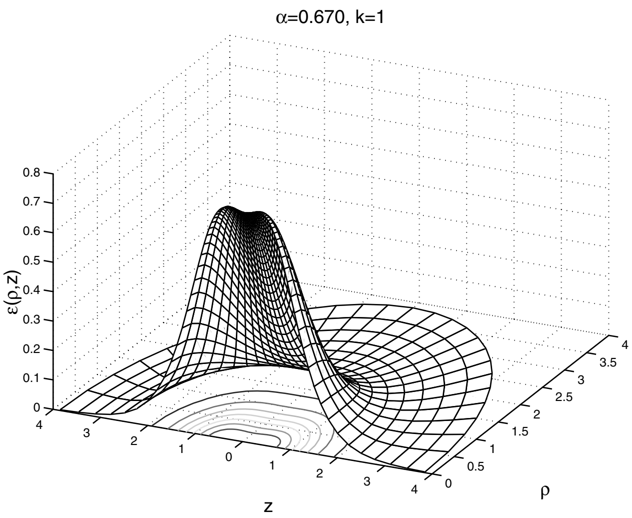

FIG. 1. The energy density $\varepsilon(\rho, z)$ is shown for the fundamental MAP solution at $\alpha_{\rm cr}^{(1)} = 0.67$.

Forming a second branch, the MAP solutions evolve smoothly backwards from $\alpha_{\rm cr}^{(1)}$ to $\alpha = 0$. In the limit $\alpha \to 0$ the mass μ diverges on this upper branch, and the locations of the monopole and antimonopole approach the origin, $\pm z_0 \to 0$, as seen in Fig. 2. At the same time the MAP solution shrinks to zero.

Rescaling the coordinate $x = \hat{x}\alpha$ and the Higgs field $\phi = \hat{\phi}/\alpha$ reveals that the axially symmetric MAP solutions approach the spherically symmetric $k = 1$ BM solution on the upper branch as $\alpha \to 0$. Consequently, the scaled mass $\hat{\mu} = \alpha\mu$ of the MAP solutions also tends to

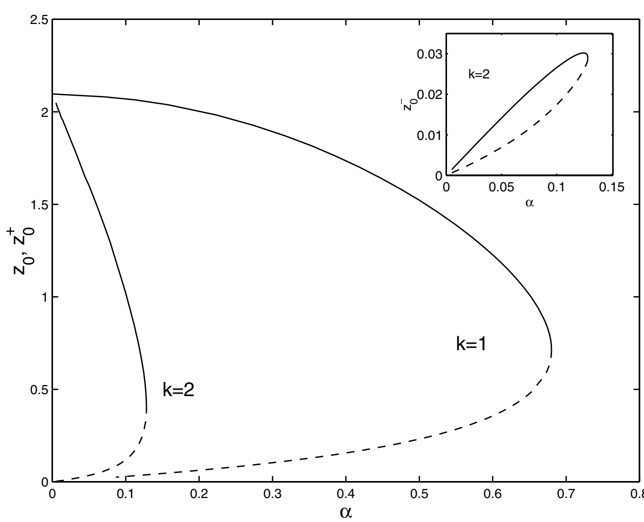

FIG. 2. For the fundamental ($k = 1$) and the first excited ($k = 2$) MAP solution the locations of the monopole, z_0 and z_0^+, respectively, are shown as functions of α. In the inset the location of the antimonopole, z_0^-, of the first excited MAP solution is shown. The solid and dashed lines correspond to the lower and upper (mass) branches, respectively.

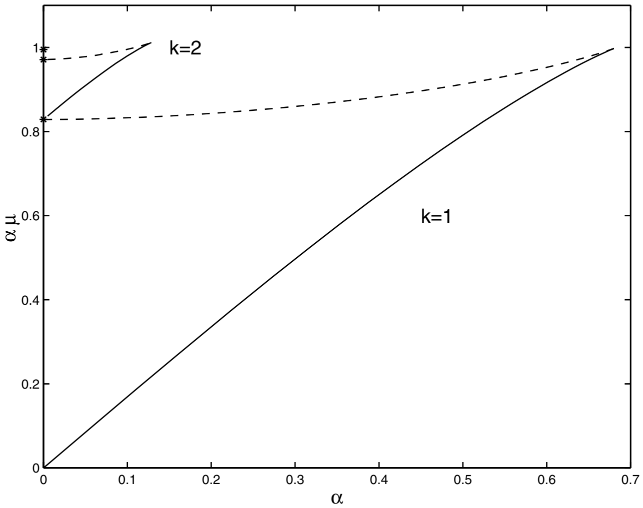

FIG. 3. The scaled mass $\hat{\mu} = \alpha\mu$ is shown as a function of α for the fundamental ($k = 1$) and the first excited ($k = 2$) MAP solution. The solid and dashed lines correspond to the lower and upper (mass) branches, respectively. The stars indicate the masses of the $k = 1, 2$, and 3 (from bottom to top) BM solutions.

the mass of the $k = 1$ BM solution, as seen in Fig. 3. On the upper branch the limit $\alpha \to 0$ thus corresponds to the limit $\eta \to 0$ (with fixed G). We note that the ansatz (3) for the gauge potential includes the spherically symmetric BM ansatz,

$$H_1 = 0, \qquad 1 - H_2 = \frac{1}{2}(1 - w),$$
$$H_3 = \frac{1}{2}\sin\theta(1 - w), \tag{7}$$
$$1 - H_4 = \frac{1}{2}\cos\theta(1 - w),$$

where w denotes the gauge field function of the BM solution.

Anticipating the existence of excited MAP solutions, linked to the BM solutions with k nodes on their upper branches, we construct the first excited MAP solution, starting from the $k = 2$ BM solution. Since the boundary conditions of the $k = 2$ BM solution differ from those of the $k = 1$ BM solution at infinity, the boundary conditions of the first excited MAP solution at infinity must be modified accordingly,

$$H_1 = H_3 = 0, \qquad H_2 = H_4 = 1, \qquad \phi_1 = \pm\cos 2\theta,$$
$$\phi_2 = \mp\sin 2\theta, \qquad f = m = l = 1. \tag{8}$$

The upper branch of the first excited MAP solutions ends at the critical value $\alpha_{\rm cr}^{(2)} = 0.128$, from where the lower branch of the excited MAP solutions evolves smoothly backwards to $\alpha = 0$. As seen in Fig. 3, in the limit $\alpha \to 0$ the scaled mass $\hat{\mu}$ approaches the mass of the $k = 2$ BM solution on the upper branch and the mass of the $k = 1$ BM solution on the lower branch.

The modulus of the Higgs field of the first excited MAP solution possesses four zeros, $\pm z_0^+$ and $\pm z_0^-$, located on the z axis, representing two monopole-antimonopole pairs. The locations of the monopole and antimonopole on the positive z axis, z_0^+ and z_0^-, respectively, are shown in Fig. 2 as functions of α, together with the node z_0 of the fundamental MAP solution. As $\alpha \to 0$, z_0^- tends to zero on both branches; in contrast, z_0^+ tends to zero only on the upper branch. On the lower branch z_0^+ tends to z_0, the location of the monopole of the fundamental MAP solution.

Inspecting the limit $\alpha \to 0$ for the first excited MAP solution on the lower branch reveals that, in terms of the radial coordinate $x = r\eta e$, the solution differs from the fundamental MAP solution on its lower branch only near the origin, where the excited MAP solution develops a discontinuity. In terms of the coordinate $\hat{x} = x/\alpha$, on the other hand, the first excited MAP solution approaches the $k = 1$ BM solution for all values of $\hat{x}$, except at infinity. Hence, the first excited MAP solution does not possess a counterpart in flat space.

Outlook.—Having constructed the fundamental and the first excited MAP solutions, the following scenario becomes evident. EYMH theory possesses a whole sequence of MAP solutions, labeled by the number of monopole-antimonopole pairs k. Each MAP solution forms two branches, merging and ending at $\alpha_{\rm cr}^{(k)}$. In the limit $\alpha \to 0$, the upper branch of the kth MAP solution always reaches the Bartnik-McKinnon solution with k nodes, while the lower branch of the kth MAP solution always reaches the Bartnik-McKinnon solution with $k - 1$ nodes, except for $k = 1$, where the flat space MAP solution is reached in the limit $\alpha \to 0$. The critical values $\alpha_{\rm cr}^{(k)}$ decrease with k, such that, as a function of α, the scaled mass $\hat{\mu}$ assumes a characteristic "Christmas tree" shape. Thus instead of the single MAP solution present in flat space, in curved space a whole tower of MAP solutions appears. An analogous pattern is encountered for gravitating Skyrmions, which are likewise linked to the BM solutions [17]. We expect the gravitating MAP solutions to be unstable like the flat space MAP solution [5].

For the gravitating monopole solutions a regular event horizon can be imposed [8–10], yielding magnetically charged black hole solutions with hair. Likewise for the MAP solutions of EYMH theory a regular event horizon can be imposed, yielding static axially symmetric and neutral black hole solutions with hair [18]. Within the framework of distorted isolated horizons the masses of these black hole solutions can be simply related to the masses of the corresponding regular solutions [18,19].

It is interesting that the spherically symmetric BM solutions of EYM theory appear in the limit $\alpha \to 0$ of the axially symmetric MAP solutions. But the BM solutions belong to a class of static axially symmetric regular solutions of EYM theory, characterized by their winding number n [14]. The MAP ansatz can be extended to include the winding number n [6,20], allowing solutions which consist of pairs of static axially symmetric multimonopoles with winding number n [2,3] and antimultimonopoles with winding number $-n$. Such multimonopole-antimultimonopole solutions will then form an analogous set of solutions as the ones encountered above but with their upper branches reaching axially symmetric EYM solutions with winding number n in the $\alpha \to 0$ limit.

But flat space also contains further interesting solutions, for instance an antimonopole-monopole-antimonopole system, with the poles located symmetrically with respect to the origin on the z axis [6].

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