Influence of Induced Interactions on the Superfluid Transition in Dilute Fermi Gases

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We calculate the effects of induced interactions on the transition temperature to the BCS state in dilute Fermi gases. For a pure Fermi system with two species having equal densities, the transition temperature is suppressed by a factor $(4e)^{1/3} \approx 2.2$, and for ν fermion species, the transition temperature is increased by a factor $(4e)^{\nu/3-1} \approx 2.2^{\nu-3}$. For mixtures of fermions and bosons the exchange of boson density fluctuations gives rise to an attractive interaction, and we estimate the increase of the transition temperature due to this effect.

The study of possible superfluidity in dilute Fermi gases has a long history stretching back to the years immediately after the development of the BCS theory [1–3]. In recent years the subject has received renewed attention for a variety of reasons. The first is that superfluidity of nucleons plays an important role in theories of neutron stars and of finite nuclei. The magnitudes of superfluid gaps are necessary input for calculations of transport properties and neutrino emission rates, and for modeling the glitches observed in the rotational periods of neutron stars. The second is theoretical interest in how properties of a Fermi system change as the strength of an attractive interaction is varied [4–6]. If the system has a two-body bound state, it will behave at low densities as a collection of diatomic molecules, while for weaker attraction it will behave as a BCS superfluid. A third reason is the possibility of observing the transition to the BCS paired state in fermion alkali atom vapors in traps [7]. Work on this topic has been spurred on by the recent success in cooling dilute alkali atom vapors to temperatures below the degeneracy temperature [8].

During the 1990's there have been a number of papers that calculate the transition temperature, or equivalently the zero temperature gap, of dilute Fermi systems [6,7,9,10]. The basic physics of these papers amounts to summing ladder diagrams with the bare fermion-fermion interaction, and expressing the result in terms of the scattering length for two-body scattering *in vacuo*. For two spin components with equal densities, these papers predict a transition temperature T_c given by

$$kT_{c0} = \frac{\gamma}{\pi} \frac{8}{e^2} E_F e^{1/N(0)U_0} \approx 0.61 E_F e^{\pi/2k_F a}, \quad (1)$$

for weak coupling [$N(0)|U_0| \ll 1$]. Here $E_F = p_F^2/2m_F$ is the Fermi energy, $p_F = \hbar k_F$ is the Fermi momentum, and m_F is the fermion mass. The quantity $N(0) = m_F k_F/(2\pi^2\hbar^2)$ is the density of states at the Fermi surface and the matrix element of the effective interaction is $U_0 = 4\pi\hbar^2 a/m_F$, where a is the scattering length for two-body scattering, which is negative for an attractive

interaction. The number γ is e^C, where $C \approx 0.577$ is Euler's constant.

The above calculation does not take into account the effects of the medium on the two-body interaction. Physically these processes, which we refer to as induced interactions, correspond to one fermion polarizing the medium, and a second fermion is then influenced by this polarization. This gives rise to an interaction between fermions analogous to the phonon-induced attraction responsible for pairing of electrons in metallic superconductors. The effects of such interactions on superfluid transition temperatures have been studied in dense systems. In liquid ^{3}He they are responsible for the ABM state being energetically favored close to the superfluid transition temperature, whereas in their absence the BW state would be the equilibrium one [11]. For neutron matter calculations predict that they suppress the superfluid gap significantly [12–14]. The corresponding effect in a dilute spin-1/2 Fermi gas was considered by Gorkov and Melik-Barkhudarov, who found that the transition temperature was suppressed by a factor $(4e)^{1/3} \approx 2.2$ compared with the result of Eq. (1) [3]. In this paper we elucidate the physical origin of the suppression, and we derive expressions for the transition temperature for fermions with a larger spin degeneracy, and for mixtures of fermions and bosons.

The transition temperature is determined by there being a solution to the linearized equation for the gap $\Delta_{\mathbf{p}}$,

$$\Delta_{\mathbf{p}} = -\sum_{\mathbf{p}'} U(\mathbf{p}, \mathbf{p}') \frac{1 - 2f^0(\xi_{\mathbf{p}'})}{2\xi_{\mathbf{p}'}} \Delta_{\mathbf{p}'}, \quad (2)$$

where $\xi_{\mathbf{p}} = p^2/2m_F - \mu$ is the energy of a fermion relative to the chemical potential μ, and $f^0(\xi) = [\exp(\xi/kT) + 1]^{-1}$ is the Fermi distribution function. The total interaction is given by

$$U(\mathbf{p}, \mathbf{p}') = U_{\text{bare}}(\mathbf{p}, \mathbf{p}') + U_{\text{ind}}(\mathbf{p}, \mathbf{p}'), \quad (3)$$

where $U_{\text{bare}}(\mathbf{p}, \mathbf{p}')$ is the bare two-body interaction, and $U_{\text{ind}}(\mathbf{p}, \mathbf{p}')$ is the induced interaction. Solving the linearized gap equation is equivalent to finding the temperature at which the T matrix obtained by summing ladder

diagrams for the repeated interaction of two particles with equal and opposite momenta and with total energy 2μ diverges. It is convenient to eliminate the bare interaction in favor of $T_0(\mathbf{p}, \mathbf{p}'; 2\mu)$, the T matrix for scattering in free space of two fermions, each with energy μ. This is given by

$$T_0(\mathbf{p}, \mathbf{p}'; 2\mu) = U_{\text{bare}}(\mathbf{p}, \mathbf{p}')$$
$$- \sum_{\mathbf{p}''} U_{\text{bare}}(\mathbf{p}, \mathbf{p}'') \frac{1}{2\xi_{\mathbf{p}''}} T_0(\mathbf{p}'', \mathbf{p}'; 2\mu). \tag{4}$$

Solving Eq. (4) for U_{bare} and inserting the result into (2) one finds

$$\Delta_{\mathbf{p}} = \sum_{\mathbf{p}'} T_0(\mathbf{p}, \mathbf{p}'; 2\mu) \frac{f^0(\xi_{\mathbf{p}'})}{\xi_{\mathbf{p}'}} \Delta_{\mathbf{p}'}$$
$$- \sum_{\mathbf{p}'} U_{\text{ind}}(\mathbf{p}, \mathbf{p}') \frac{1 - 2f^0(\xi_{\mathbf{p}'})}{2\xi_{\mathbf{p}'}} \Delta_{\mathbf{p}'}$$
$$+ \sum_{\mathbf{p}', \mathbf{p}''} T_0(\mathbf{p}, \mathbf{p}''; 2\mu) \frac{1}{2\xi_{\mathbf{p}''}} U_{\text{ind}}(\mathbf{p}'', \mathbf{p}')$$
$$\times \frac{1 - 2f^0(\xi_{\mathbf{p}'})}{2\xi_{\mathbf{p}'}} \Delta_{\mathbf{p}'}. \tag{5}$$

Let us first consider the transition temperature in the absence of induced interactions. The factor $f^0(\xi_{\mathbf{p}'})/\xi_{\mathbf{p}'}$ falls off rapidly for momenta greater than the Fermi momentum. According to the standard effective range expansion, the on-shell T matrix for small p and p' is its value for $p = p' = 0$ and zero energy, $T_0(0, 0; 0) = U_0$, plus terms of order p^2. The latter terms produce higher-order contributions than those from the induced interaction, and will be neglected here. Thus in Eq. (5) the gap may be put equal to a constant for momenta less than or of order the Fermi momentum, and the temperature at which there is a nontrivial solution to the equation is given by

$$U_0 \int_0^\infty \frac{p^2 dp}{2\pi^2 \hbar^3} \frac{1}{\xi_p} \frac{1}{e^{\xi_p/kT_{c0}} + 1} = 1. \tag{6}$$

On performing the integration one arrives at Eq. (1).

We now include the effect of induced interactions, which we shall assume to be small compared with the interaction of two fermions *in vacuo*. In Eq. (5) the last term represents a final-state interaction. The range of the induced interaction is of order the spacing between fermions or, for the phonon-induced interaction in mixtures, the coherence length. Both these lengths are large compared with the magnitude of the fermion-fermion scattering length, and therefore in the region where the induced interaction is important the wave function of two fermions in the presence of the bare interaction, which is equal to $1 - a/r$, is essentially equal to unity, and final-state effects are neg-

ligible. One then finds that the transition temperature is given to leading order in U_{ind} by

$$kT_c = \frac{\gamma}{\pi} \frac{8}{e^2} E_F e^{1/N(0)(U_0 + \bar{U}_{\text{ind}})}, \tag{7}$$

where

$$\bar{U}_{\text{ind}} = \int_{-1}^1 \frac{d\cos\theta}{2} U_{\text{ind}}(p_F \hat{\mathbf{n}}, p_F \hat{\mathbf{n}}'), \tag{8}$$

is the average of the induced interaction over the Fermi surface. The angle between the two momenta is denoted by θ, i.e., $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' = \cos\theta$. Note that the frequency dependence of the induced interaction does not enter, only its value at the Fermi surface. When the induced interaction becomes comparable to the bare interaction, the effects of the frequency dependence of the interaction will become important.

We now apply this result to Fermi systems, and we begin by considering a Fermi gas with two internal degrees of freedom, denoted in Fig. 1 by up and down arrows. We shall take the densities of the two components to be equal, since this gives the largest gap. The leading contributions to the induced interaction are represented diagrammatically in Fig. 1. One may ask why they should have any effect at all in this limit, since they are formally of higher order in the density than the leading term. To understand this, consider the process shown in Fig. 1(a), which is the screening of the interaction between two fermions by the other fermions. This is of order the density of states times the square of the effective two-body interaction, and is therefore of order $k_F a U_0$. If one alters the effective interaction in the expression (1) by an amount of relative order $k_F a$, it is easy to see that the gap is multiplied by a constant factor, as the calculation of Gorkov and Melik-Barkhudarov demonstrated explicitly [3]. For
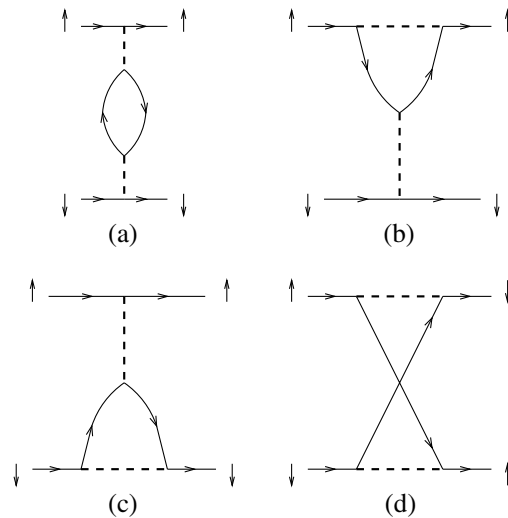

FIG. 1. Diagrams for the induced interactions between two fermions in different internal states to second order in the effective interaction.

a contact interaction, the contributions of the diagrams in Fig. 1 give

$$U_{\text{ind}}(p_F\hat{\mathbf{n}}, p_F\hat{\mathbf{n}}') = U_0^2 L(q), \qquad (9)$$

where

$$L(q) = N(0)\left(\frac{1}{2} + \frac{(1 - \eta^2)}{4\eta} \ln\left|\frac{1 + \eta}{1 - \eta}\right|\right) \qquad (10)$$

is the static Lindhard function and $\eta = q/2k_F$. Here, $\hbar q$ is equal to the magnitude of $\mathbf{p} + \mathbf{p}'$ which, as both particles are at the Fermi surface, is related to the scattering angle θ by $q^2 = 2k_F^2(1 + \cos\theta)$. Carrying out the integration over angles in Eq. (8) gives

$$\bar{U}_{\text{ind}} = U_0^2 N(0) \frac{1 + 2\ln 2}{3}, \qquad (11)$$

and by insertion of this expression in Eq. (7) we find

$$T_c = \frac{1}{(4e)^{1/3}} T_{c0}, \qquad (12)$$

a result implicit in Ref. [3]. We therefore arrive at the striking conclusion that induced interactions reduce the transition temperature by a factor $(4e)^{1/3} \approx 2.2$ even in the low density limit. It is interesting that calculations of the superfluid gap in neutron matter at densities at which the low-density limit is inapplicable indicate that induced interactions reduce the maximum of the gap as a function of density by a comparable factor [12–14], but this is a coincidence.

The physics of the suppression is best examined by expressing the result in terms of the amplitudes for exchange of density and spin fluctuations. In terms of these one finds for the interaction in the spin-singlet channel for the pair of fermions

$$U_{\text{ind}}(q) = [U_s(q) - 3U_t(q)]/2, \qquad (13)$$

where U_s and U_t are the amplitudes for exchange of spin-singlet particle-hole pairs (density fluctuations) and spin-triplet ones (spin fluctuations), respectively. The leading contributions to the amplitudes are $U_s = U_t = -U_0^2 L(q)$, and therefore $U_{\text{ind}}(q) = U_0^2 L(q)$. One way of expressing this is to note that the contributions from the diagrams in Figs. 1(a)–1(c), which are due to exchange of spin and density fluctuations with spin projection zero, cancel, leaving the contribution from exchange of spin fluctuations with spin projection ± 1, Fig. 1(d). The important process is therefore the suppression of superfluidity by exchange of spin fluctuations, an effect familiar in metallic superconductors and liquid ^{3}He [15].

The result above may easily be generalized to ν fermion species. For simplicity we assume that the two-body interaction is independent of the species, and is diagonal in the species labels. This is true if the interaction is spin independent and depends only on the distance between the two fermions. The contribution from the diagram in Fig. 1(a) is proportional to ν, because all species contribute to the closed loop, while the other diagrams in Fig. 1 are independent of the number of species, since there are no

fermion loops. Thus the contribution from all diagrams is the same as for two species, except that the first term is multiplied by ν, and their sum is proportional to $\nu - 3$. The transition temperature is given by

$$T_c = (4e)^{\nu/3 - 1} T_{c0}, \qquad (14)$$

and therefore for four or more components the transition temperature is *increased,* rather than decreased.

The results above are not directly applicable to dilute nuclear matter because the nuclear force is strongly dependent on isospin: the neutron-neutron scattering length is approximately -18.8 fm, while the neutron-proton one is 5.4 fm. As a consequence, the induced interaction between two neutrons, say, due to excitation of neutron particle-hole pairs is of order $(18.8/5.4)^2 \approx 12$ times stronger than that due to excitation of proton pairs, and therefore the latter contribution may be neglected to a first approximation. The fact that induced interactions are important for pairing even at very low densities in bulk systems indicates the need to investigate how significant these effects are in the outer parts of finite nuclei.

We now consider adding bosons of mass m_B to a Fermi gas with two species. The induced interaction between fermions then contains a contribution due to the exchange of boson density fluctuations, analogous to the phonon-induced interaction between electrons in metals and to the induced interaction between ^{3}He atoms in dilute solutions of ^{3}He in ^{4}He. It is represented diagrammatically in Fig. 2 and is given by [16,17]

$$U_{\text{ind}} = U_{BF}^2 \chi(q, \omega), \qquad (15)$$

where $U_{BF} = 4\pi\hbar^2 a_{BF}/m_{BF}$ [with $m_{BF} = 2m_F m_B/(m_F + m_B)$] is the effective interaction between a boson and a fermion, and $\chi(q, \omega)$ is the density-density response function for the bosons as a function of the energy transfer $\hbar\omega$. For a dilute Bose gas the response function at zero temperature is given by the result of using the Bogoliubov approximation, and is

$$\chi(q, \omega) = \frac{n_B \hbar^2 q^2/m_B}{(\hbar\omega)^2 - \epsilon_q^0(\epsilon_q^0 + 2n_B U_{BB})}, \qquad (16)$$

where $U_{BB} = 4\pi\hbar^2 a_{BB}/m_B$ is the boson-boson effective interaction and $\epsilon_q^0 = \hbar^2 q^2/2m_B$. For particles at the Fermi surface the energy transfer is zero, and therefore the
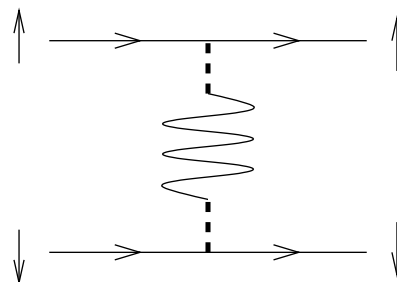

FIG. 2. Diagram for the phonon-induced interaction between two fermions. The dashed line is the fermion-boson interaction, and the wavy line is a phonon in the boson gas.
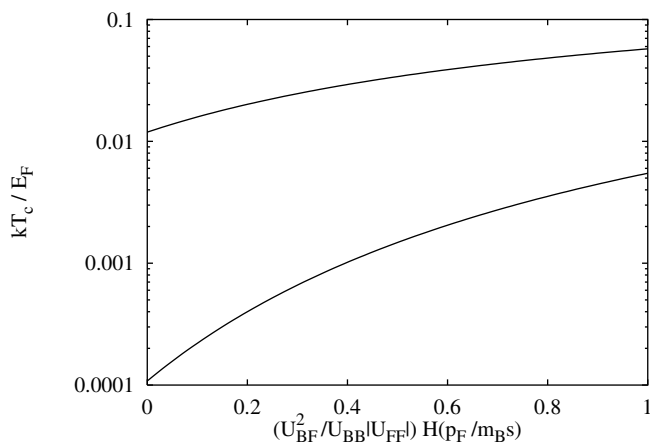
FIG. 3. Transition temperature as a function of the parameter $(U_{BF}^2/U_{BB}|U_{FF}|)H(p_F/m_B s)$, for $k_F|a| = 0.2$ (lower curve) and $k_F|a| = 0.5$ (upper curve).

induced interaction relevant for calculating the transition temperature is

$$U_{\mathrm{ind}}(q,0) = -\frac{U_{BF}^2}{U_{BB}} \frac{1}{1 + (\hbar q/2m_B s)^2}, \qquad (17)$$

where $s = (n_B U_{BB}/m_B)^{1/2}$ is the sound velocity in the boson gas. The average over the Fermi surface of the boson-induced interaction between fermions that enters the equation for the gap is

$$\bar{U}_{\mathrm{ind}} = -\frac{U_{BF}^2}{U_{BB}} H(p_F/m_B s), \qquad (18)$$

where $H(x) = \ln(1 + x^2)/x^2$. Observe that this result is independent of the density of fermions in the limit of low fermion densities. This is in contrast to the induced interaction due to fermion-fermion interactions alone, which is proportional to p_F. If we again assume that $\bar{U}_{\mathrm{ind}}$ is small compared with the fermion-fermion interaction, the transition temperature is given by Eq. (7), where the induced interaction includes terms due to fermion-fermion interactions as well as fermion-boson interactions. In Fig. 3 this is plotted as a function of the dimensionless quantity $(U_{BF}^2/U_{BB}|U_{FF}|)H(p_F/m_B s)$. (To make the notation uniform we here denote the fermion-fermion interaction, which we assume to be attractive, by U_{FF} rather than U_0 as we did in the earlier part of the Letter.) The parameter $U_{BF}^2/U_{BB}|U_{FF}|$ is a measure of the importance of the induced interactions compared with the direct one. Since the function H decreases monotonically from its maximum value of 1 for $x = 0$, the largest relative effects are achieved for small fermion densities. As Fig. 3 shows, the increase in T_c due to the presence of the bosons can be considerable.

Let us now examine a specific example, a mixture of the fermionic atoms ^{6}Li with the bosonic atoms ^{87}Rb. For the boson-boson scattering length we use $a_{BB} = 109a_0$,

a_0 being the Bohr radius, and for the fermion-fermion one $a = -2160a_0$. The boson-fermion scattering length is not known. If we choose the order-of-magnitude estimate $|a_{BF}| = 100a_0$ [7], $U_{BF}^2/U_{BB}|U_{FF}|$ is only 0.18. For $n_B = 10^{14}$ cm^{-3} the increase in T_c then amounts to factors of 2.7 and 1.3 for $k_F|a| = 0.2$ and $k_F|a| = 0.5$, respectively. If, however, $|a_{BF}|$ were twice as large, these factors would change to 17.6 and 2.2. This indicates the importance of obtaining information about boson-fermion scattering lengths.

Finally we remark that more detailed calculations are required to take into account induced interactions when they become comparable in magnitude to the bare interaction. The details of the frequency dependence of the induced interactions will then become important. We note that corrections to the effective mass of a fermion and to the renormalization constant give corrections to T_c of higher order than those considered here.

[1] L. van Hove, Physica (Amsterdam) **25**, 849 (1959).
[2] V. J. Emery, Nucl. Phys. **19**, 154 (1960); V. J. Emery and A. M. Sessler, Phys. Rev. **119**, 248 (1960).
[3] L. P. Gorkov and T. K. Melik-Barkhudarov, Sov. Phys. JETP **13**, 1018 (1961).
[4] A. J. Leggett, in *Modern Trends in the Theory of Condensed Matter,* edited by A. Pekalski and R. Przystawa (Springer-Verlag, Berlin, 1980); J. Phys. (Paris), Colloq. **41**, C7-19 (1980).
[5] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. **59**, 195 (1985).
[6] C. A. R. Sá de Melo, M. Randeria, and J. R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
[7] H. T. C. Stoof, M. Houbiers, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. **76**, 10 (1996).
[8] B. DeMarco and D. S. Jin, Science **285**, 1703 (1999); M. J. Holland, B. DeMarco, and D. S. Jin, Phys. Rev. A **61**, 053610-1 (2000).
[9] T. Papenbrock and G. F. Bertsch, Phys. Rev. C **59**, 2052 (1999).
[10] V. A. Khodel, V. V. Khodel, and J. W. Clark, Nucl. Phys. A **598**, 390 (1996).
[11] D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor and Francis, London, 1990), p. 123.
[12] J. M. C. Chen, J. W. Clark, R. D. Davé, and V. V. Khodel, Nucl. Phys. A **555**, 59 (1993).
[13] J. Wambach, T. Ainsworth, and D. Pines, Nucl. Phys. A **555**, 128 (1993).
[14] H.-J. Schulze, J. Cugnon, A. Lejeune, M. Baldo, and U. Lombardo, Phys. Lett. B **375**, 1 (1996).
[15] N. F. Berk and J. R. Schrieffer, Phys. Rev. Lett. **17**, 433 (1966).
[16] L. Viverit, C. J. Pethick, and H. Smith, Phys. Rev. A **61**, 053605-1 (2000).
[17] M. Bijlsma, B. A. Heringa, and H. T. C. Stoof, Phys. Rev. A **61**, 053601-1 (2000).