Localizing Gravity on a Stringlike Defect in Six Dimensions

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We present a metric solution in six dimensions, where gravity is localized on a four-dimensional singular stringlike defect. The corrections to four-dimensional gravity from the bulk continuum modes are suppressed by $\mathcal{O}(1/r^3)$. No tuning of the bulk cosmological constant to the brane tension is required in order to cancel the four-dimensional cosmological constant.

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It is an old idea that spacetime may have more than four dimensions, with extra coordinates being unobservable at available energies. A first possibility arises in Kaluza-Klein-type theories (see, e.g., Ref. [1], and references therein), where the D-dimensional metric has the form

$$ds^2 = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu - \gamma_{ab}(x^a)dx^a dx^b. \quad (1)$$

Here $g_{\mu\nu}$ is the metric of our four-dimensional world, while γ_{ab} is the metric associated with $D - 4$ (small, with a size M^{-1}) compact extra dimensions. The compactness of extra dimensions makes them unobservable at energies $E < M$, and manifests itself in the existence of an infinite tower of states with four-dimensional masses $\sim M$.

In fact, the Kaluza-Klein metric is not the most general metric consistent with Poincaré invariance in four dimensions. Its generalization was proposed in [2], and is given by

$$ds^2 = \sigma(x^a)g_{\mu\nu}(x^\mu)dx^\mu dx^\nu - \gamma_{ab}(x^a)dx^a dx^b, \quad (2)$$

where $\sigma(x^a)$ is a conformal factor depending on the extra coordinates only. A number of specific solutions of the Einstein equations in six-dimensional (6D) spacetime with a positive 6D cosmological constant were found in [2], leading to noncompact extra dimensions while still leaving them unobservable at low energies.

Yet another idea leading to noncompact extra dimensions was suggested in [3–5]. In this case the four dimensions of our world were identified with the internal space of topological defects residing in a higher-dimensional spacetime (e.g., a domain wall in 5D, string in 6D, monopole in 7D, instanton in 8D, etc.). In these types of backgrounds, as a rule, there are fermionic and scalar zero modes that can be associated with the four-dimensional particles that we observe. At that time it was not clear how to localize the gauge fields and gravity on topological defects in order to make the whole construction realistic.

The solitons of string theory—D-branes—provide a natural framework for the localization of gauge and matter fields on the world volume of the branes [6]. In field theory language the branes can be associated with topological defects. Moreover, in Ref. [7] it was discovered that gravity could be localized on the 3-brane domain wall in 5D spacetime. A normalizable graviton zero mode residing on the brane correctly reproduces 4D gravity, while the continuum spectrum of 5D gravitons, living in the bulk, gives only a small correction $\mathcal{O}(1/r^2)$ to Newton's law at large distances [7]. The metric of the corresponding 5D spacetime has the general structure of Eq. (2).

The aim of this paper is to see what happens with gravity around a 3-brane of a specific structure (local string defect in field theory language) in 6D spacetime with a negative cosmological constant. In fact, a regular solution of the Einstein equations in this case for an empty space follows immediately from [2], but does not give any possibility of compactification. However, the existence of a brane with positive tension changes the situation and we find a solution which is very similar to that of Ref. [7]. In contrast to the 5D case, there is no fine-tuning of the cosmological constant in the bulk to the tension of the brane (the origin of this difference is that the 1D space in the domain wall scenario is flat, while the 2D space around the string defect can be curved). Similar to the solution in Ref. [7], there is a normalizible graviton zero mode attached to the stringlike defect, and the contribution of bulk gravitons is suppressed, leading to $\mathcal{O}(1/r^3)$ violations of Newton's law. A hierarchy between the four-dimensional Planck scale and the Planck scale in 6D can be obtained, leading to a solution of the gauge hierarchy problem similar to that of Ref. [8].

Other solutions obtained with two transverse dimensions include a generalization of the original 5D domain wall setup to the case of parallel brane sources [9], and the case of global string defects [10–12]. Furthermore, a class of radially symmetric solutions was considered in [13,14].

Let us begin with the details of our solution. In 6D the Einstein equations with a bulk cosmological constant Λ and stress-energy tensor T_{AB} are

$$R_{AB} - \frac{1}{2} g_{AB}R = \frac{1}{M_6^4} (\Lambda g_{AB} + T_{AB}), \quad (3)$$

where M_6 is the six-dimensional reduced Planck scale. We will assume that there exists a solution that respects 4D Poincaré invarance. A six-dimensional metric satisfying this ansatz is

$$ds^2 = \sigma(\rho)g_{\mu\nu}dx^\mu dx^\nu - d\rho^2 - \gamma(\rho)d\theta^2, \quad (4)$$

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where the metric signature is $(+,-,-,-)$. For the two extra spatial dimensions we have introduced polar coordinates (ρ,θ), where $0 \leq \rho < \infty$ and $0 \leq \theta < 2\pi$. With our metric ansatz (4), the general expression for the four-dimensional reduced Planck scale M_P, expressed in terms of M_6, is

$$M_P^2 = 2\pi M_6^4 \int_0^\infty d\rho\, \sigma\sqrt{\gamma}\,. \tag{5}$$

The nonzero components of the stress-energy tensor T_B^A are assumed to be

$$T_\nu^\mu = \delta_\nu^\mu f_0(\rho), \quad T_\rho^\rho = f_\rho(\rho), \quad \text{and} \quad T_\theta^\theta = f_\theta(\rho)\,, \tag{6}$$

where we have introduced three source functions f_0, f_ρ, and f_θ, which depend only on the radial coordinate ρ. By using the cylindrically symmetric metric ansatz (4) and the stress-energy tensor (6), the Einstein equations become

$$\begin{aligned} \frac{3}{2}\frac{\sigma''}{\sigma} + \frac{3}{4}\frac{\sigma'}{\sigma}\frac{\gamma'}{\gamma} - \frac{1}{4}\frac{\gamma'^2}{\gamma^2} + \frac{1}{2}\frac{\gamma''}{\gamma} &= -\frac{1}{M_6^4}[\Lambda + f_0(\rho)] + \frac{1}{M_P^2}\frac{\Lambda_{\rm phys}}{\sigma}\,,\\ \frac{3}{2}\frac{\sigma'^2}{\sigma^2} + \frac{\sigma'}{\sigma}\frac{\gamma'}{\gamma} &= -\frac{1}{M_6^4}[\Lambda + f_\rho(\rho)] + \frac{1}{M_P^2}\frac{2\Lambda_{\rm phys}}{\sigma}\,,\\ 2\frac{\sigma''}{\sigma} + \frac{1}{2}\frac{\sigma'^2}{\sigma^2} &= -\frac{1}{M_6^4}[\Lambda + f_\theta(\rho)] + \frac{1}{M_P^2}\frac{2\Lambda_{\rm phys}}{\sigma}\,, \end{aligned} \tag{7}$$

where the prime denotes differentiation $d/d\rho$. The constant $\Lambda_{\rm phys}$ represents the physical four-dimensional cosmological constant, where

$$R_{\mu\nu}^{(4)} - \frac{1}{2}g_{\mu\nu}R^{(4)} = \frac{1}{M_P^2}\Lambda_{\rm phys}g_{\mu\nu}\,. \tag{8}$$

By eliminating two of the equations in (7), the sources can be related by the following equation:

$$f_\rho' = 2\frac{\sigma'}{\sigma}(f_0 - f_\rho) + \frac{1}{2}\frac{\gamma'}{\gamma}(f_\theta - f_\rho)\,. \tag{9}$$

In the absence of source terms, the discussion of solutions to this coupled system of differential equations for arbitrary values of $\Lambda_{\rm phys}$ and $\Lambda > 0$ can be found in [2]. However, the case $\Lambda < 0$ was not considered there because the vacuum solutions lead to noncompact transverse spaces, and therefore, using (5), one cannot obtain a finite value of the Planck scale. Here we propose adding singular source terms in order to obtain a transverse space with finite volume (which leads to a finite four-dimensional Planck scale). Thus, the system of Eqs. (7) and (9) describes the generalization of the setup considered in [2], to the case where source terms are included. Similar equations of motion in the global string context were also considered in Refs. [10–12].

Specifically, we will assume that there is a 3-brane at the origin $\rho = 0$ which is a four-dimensional local stringlike topological defect in the six-dimensional spacetime, and has a nonzero stress-energy tensor T_B^A parametrized by (6). For example, one may think of the Nielsen-Olesen string solution in the 6D Abelian Higgs model. The source functions describe a continuous matter distribution within the core of radius ϵ and vanish for $\rho > \epsilon$. At the origin we will require that our solution satisfies the boundary conditions

$$\sigma'|_{\rho=0} = 0, \quad (\sqrt{\gamma})'|_{\rho=0} = 1, \quad \text{and} \quad \gamma|_{\rho=0} = 0\,. \tag{10}$$

We have set $\sigma(0) = 1$, since the arbitrary integration constant corresponds to an overall rescaling of the coordinates x^μ. Following [15], we can integrate over the disk of small radius ϵ containing the 3-brane, and define various components of the string tension per unit length as

$$\mu_i = \int_0^\epsilon d\rho\, \sigma^2\sqrt{\gamma}\, f_i(\rho)\,. \tag{11}$$

where $i = 0, \rho, \theta$. Using the system of Eqs. (7), we obtain the following boundary conditions:

$$\sigma\sigma'\sqrt{\gamma}|_0^\epsilon = -\frac{1}{2M_6^4}(\mu_\rho + \mu_\theta)\,, \tag{12}$$

and

$$\sigma^2(\sqrt{\gamma})'|_0^\epsilon = -\frac{1}{M_6^4}\left(\mu_0 + \frac{1}{4}\mu_\rho - \frac{3}{4}\mu_\theta\right), \tag{13}$$

where it is understood that the limit $\epsilon \to 0$ is taken. By analogy with string defects in four dimensions, $\mu_\rho + \mu_\theta$ can be referred to as the Tolman mass (per unit length) [16]. Its nonzero value in four dimensions gives rise to the Melvin branch for local string defects [17]. Similarly, the analogous equation of (13) in four dimensions is related to the string angular deficit [17]. Thus, with these general conditions, any metric solution to the Einstein equations with sources will lead to nontrivial relationships between the components of the string tension per unit length.

Let us now restrict ourselves to the case where the four-dimensional cosmological constant $\Lambda_{\rm phys} = 0$, and look for a solution of the form

$$\sigma(\rho) = e^{-c\rho}\,. \tag{14}$$

Then, a solution to the coupled set of Eqs. (7) can be found with $\gamma(\rho) = R_0^2\sigma(\rho)$ and

$$c = \sqrt{\frac{2}{5}\frac{(-\Lambda)}{M_6^4}}\,, \tag{15}$$

where R_0 is an arbitrary length scale that can be fixed from Eqs. (12) and (13). Clearly, the negative exponential solution (14) requires that $\Lambda < 0$. If we now demand that the solution (15) is consistent with the boundary conditions (12) and (13), the components of the string tension per unit length must satisfy

$$\mu_0 = \mu_\theta + M_6^4, \tag{16}$$

where μ_ρ remains undetermined. In fact, choosing $\mu_\rho = 0$ gives

$$\mu_\theta = 2R_0 M_6^4 c\,. \tag{17}$$

Thus, as long as sources are introduced at the origin $\rho = 0$ satisfying (16), we obtain a flat Poincaré invariant solution in four dimensions. Since the solution is already valid for $\Lambda_{\rm phys} = 0$, there is no need to tune the brane tension to the bulk cosmological constant Λ, as in the case [7]. However, there is still a tuning in order to satisfy (16).

Having found a solution with a finite volume transverse space, the four-dimensional reduced Planck scale now becomes

$$M_P^2 = 2\pi R_0 M_6^4 \int_0^\infty d\rho\, \sigma^{3/2} = \frac{5\pi}{3}\,\frac{\mu_\theta}{-\Lambda}\, M_6^4\,, \tag{18}$$

where we have used the relation (17). The inequality $M_6 \ll M_P$ is possible by adjusting the string tension or the bulk cosmological constant, and thus could lead to a solution of the gauge hierarchy problem along the lines of [8].

In order to see that gravity is only localized on the 3-brane, let us now consider the equations of motion for the linearized metric fluctuations. We will concentrate only on the spin-2 modes and neglect the scalar modes, which need to be taken into account together with the bending of the brane [18]. The vector modes are massive as follows from a simple modification of the results in Ref. [19]. For a fluctuation of the form $h_{\mu\nu}(x,z) = \Phi(z) h_{\mu\nu}(x)$ where $z = (\rho, \theta)$ and $\partial^2 h_{\mu\nu}(x) = m_0^2 h_{\mu\nu}(x)$, we can separate the variables by defining $\Phi(z) = \sum_{lm} \phi_m(\rho) e^{il\theta}$. The radial modes satisfy the equation [19]:

$$-\frac{1}{\sigma\sqrt{\gamma}}\,\partial_\rho[\sigma^2\sqrt{\gamma}\,\partial_\rho \phi_m] = m^2 \phi_m\,, \tag{19}$$

where $m_0^2 = m^2 + l^2/R_0^2$ contains the contributions from the orbital angular momentum l. The differential operator (19) is self-adjoint provided that we impose the boundary conditions

$$\phi_m'(0) = \phi_m'(\infty) = 0\,, \tag{20}$$

where the modes ϕ_m satisfy the orthonormal condition

$$\int_0^\infty d\rho\, \sigma\sqrt{\gamma}\, \phi_m^* \phi_n = \delta_{mn}\,. \tag{21}$$

Using the specific solution (15), the differential operator (19) becomes

$$\phi_m'' - \frac{5}{2}\, c\,\phi_m' + m^2 e^{c\rho} \phi_m = 0\,. \tag{22}$$

This equation is the same as that obtained for the 5D domain wall solution [7], except that the coefficient of the first-derivative term is 2 instead of $5/2$. This difference is due to the extra spatial coordinate in the transverse space. When $m = 0$ we clearly see that $\phi_0(\rho) = \text{const}$ is a solution. Since the modes satisfy the orthonormal condition

$$R_0 \int_0^\infty d\rho\, e^{-(3/2)c\rho}\, \phi_m^* \phi_n = \delta_{mn}\,, \tag{23}$$

a wave function in flat space can be defined as

$$\psi_m = e^{-(3/4)c\rho}\, \phi_m\,. \tag{24}$$

Thus the zero-mode wave function becomes

$$\psi_0(\rho) = \sqrt{\frac{3c}{2R_0}}\, e^{-(3/4)c\rho}, \tag{25}$$

which shows that the zero-mode tensor fluctuation is localized near the origin $\rho = 0$ and is normalizable.

The contribution from the nonzero modes will modify Newton's law on the 3-brane. In order to calculate this contribution we need to obtain the wave function for the nonzero modes at the origin. The nonzero mass eigenvalues can be obtained by imposing the boundary conditions (20) on the solutions of the differential equation (22). The solutions of (22) are

$$\phi_m(\rho) = e^{(5/4)c\rho}\left[C_1 J_{5/2}\left(\frac{2m}{c}\, e^{(c/2)\rho}\right) + C_2 Y_{5/2}\left(\frac{2m}{c}\, e^{(c/2)\rho}\right)\right], \tag{26}$$

where C_1, C_2 are constants and $J_{5/2}, Y_{5/2}$ are Bessel functions which can be expressed in terms of elementary functions. In the limit that $\rho \to \infty$, the solutions for nonzero m grow exponentially. One way to regulate this behavior is to introduce a finite radial distance cutoff $\rho_{\max}$. Then imposing the boundary conditions (20) at $\rho = \rho_{\max}$ (instead of $\rho = \infty$) will lead to a discrete mass spectrum, where for sufficiently large integer n we obtain

$$m_n \simeq c\left(n - \frac{1}{2}\right)\frac{\pi}{2}\, e^{-(c/2)\rho_{\max}}. \tag{27}$$

With this discrete mass spectrum we find that, in the limit of vanishing mass m_n,

$$\phi_{m_n}^2(0) = \frac{4}{cR_0}\, m_n^2 e^{-(c/2)\rho_{\max}}. \tag{28}$$

On the 3-brane the gravitational potential between two point masses m_1 and m_2 will receive a contribution from the discrete nonzero modes given by

$$\Delta V(r) \simeq G_N\, \frac{m_1 m_2}{r} \sum_n e^{-m_n r}\, \frac{8}{3c^2}\, m_n^2 e^{-(c/2)\rho_{\max}}, \tag{29}$$

where G_N is Newton's constant. In the limit that $\rho_{\max} \to \infty$, the spectrum becomes continuous and the discrete sum is converted into an integral. Thus the contribution to the gravitational potential becomes

$$\Delta V(r) \simeq \frac{16 G_N}{3\pi c^3} \frac{m_1 m_2}{r} \int_0^\infty dm\, m^2 e^{-mr}, \tag{30}$$

$$= \frac{32 G_N}{3\pi c^3} \frac{m_1 m_2}{r^4}. \tag{31}$$

Thus we see that the correction to Newton's law from the bulk continuum states grows like $1/r^3$. This correction is more suppressed than in 5D, because now the gravitational field of the bulk continuum modes spreads out in one extra dimension and so their effect on the 3-brane is weaker.

Some remarks are now in order:

(i) If different components of the brane tension do not satisfy Eq. (16), a more general solution to the system of Eqs. (7) can be found along the lines of Ref. [2]. By using the parametrization $\sigma = z^{4/5}$, and $\gamma = \alpha^2 (z')^2 z^{-6/5}$ [with $\alpha = 4R_0/(5c)$], the general solution can be written as

$$z(\rho) = \exp\left(-\frac{5}{4} c\rho\right) + 2\beta \sinh\left(-\frac{5}{4} c\rho\right), \tag{32}$$

where $\beta = 0$ corresponds to the case (14). The general condition for the brane tension components now becomes

$$\mu_0 - \mu_\theta = \beta(\beta + 1)\left(\frac{3}{2}\mu_\theta - \frac{5}{2}\mu_\rho - 4\mu_0\right) + (1 + 2\beta)^2 M_6^4. \tag{33}$$

The choice of $\beta < 0$ does not lead to any compactification because σ diverges at large ρ. However, $\beta > 0$ leads to noncompact spaces defined for a finite interval $0 < \rho < \frac{2}{5c}\log(\frac{1+\beta}{\beta})$ of the type discussed in [2] that may be used as a description of four-dimensional space.

(ii) The metric solution that we have found can also be written in the form

$$ds^2 = z^2 g_{\mu\nu} dx^\mu dx^\nu - R_0^2 z^2 d\theta^2 - \frac{4}{c^2 z^2} dz^2. \tag{34}$$

where $z = \exp(-\frac{c}{2}\rho)$. In this way we see that the origin $\rho = 0$ is now mapped to $z = 1$. The singular source is spread around the circumference of a disk of radius R_0. This suggests that the 3-brane at the origin $\rho = 0$ can be interpreted as a wrapped 4-brane, where all angular points θ are identified. In other words, by denoting the wrapped 4-brane by $\mathcal{M}_4$, the 3-brane corresponds to $\mathcal{M}_4/S^1$. While we have given the explicit solution in six dimensions, our solution can be generalized, and presumably similar solutions exist at the core of topological defects in higher dimensions where, for $n \geq 2$ transverse dimensions, the 3-brane can be identified with $\mathcal{M}_{n+2}/S^{n-1}$, where $\mathcal{M}_{n+2}$ has $n - 1$ coordinates spherically wrapped. Again, the corrections to 4D gravity on the 3-brane are expected to be small since the bulk continuum modes live in a higher-dimensional space and, by Gauss's law, the effects on the 3-brane are suppressed.

(iii) It is also interesting to study whether our solution (or its generalization in higher dimensions) can be realized in an effective supergravity theory. This would be one step towards embedding the scenario in string theory.

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