Chaotic Transition of Random Dynamical Systems and Chaos Synchronization by Common Noises

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We studied the mechanism behind the connection between the transition to chaos of random dynamical systems and the synchronization of chaotic maps driven by external common noises. Near the chaotic transition, the spatial size of random dynamical systems shows an extreme intermittent behavior. By calculating the scaling exponents, we have found that the origin of this intermittent behavior is *on-off intermittency*. This led us to conclude that chaotic transitions through on-off intermittency can be regarded as a *route* for random dynamical systems. To clarify this argument, a two-dimensional random dynamical system and two coupled logistic maps driven by external common noises were analyzed.

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The transition to chaos of random dynamical systems was studied by Yu, Ott, and Chen (YOC) [1] a decade ago. Their study was motivated by the fact that the transition to chaos in deterministic systems usually occurs through several specific routes, such as period doubling, intermittency, quasiperiodicity, and crises. Their question was: *Then, is there any specific route through which random dynamical systems would go during chaotic transition?* As the first step they considered the motion of floating particles on a surface of incompressible fluid. One of the main results of their study was that there is a transition with variation of a parameter from a state where initial distribution of particles eventually coalesces to a point to a state where particles are distributed on a fractal, so called a *snapshot attractor.* They also observed that, near the critical point, the time dependence of the spatial dimension of the snapshot attractor can exhibit an extreme form of *temporally intermittent bursting.* It resembles on-off intermittency which was first reported by Platt, Spiegel, and Tresser [2] in 1993. Heagy, Platt, and Hammel [3] commented that the snapshot attractor can undergo a form of intermittent behavior that is similar to on-off intermittency but that the dimensional distributions of the snapshot attractor are different from the laminar phase distributions of on-off intermittency.

In close connection with the matter, the topic of synchronization of chaotic maps has been one of the most active fields of research in recent years because of the importance of its practical applications in many different disciplines, e.g., secure communications, controlling optical chaos, etc. Especially one of the schemes of chaotic synchronization, i.e., synchronization by common external noises, was a rather controversial subject since Maritan and Banavar (MB) [4] first reported about it in 1994. Pikovsky [5] immediately pointed out that the synchronization effects which MB observed are just the results of finite precisions of the computer analysis. Then Longa *et al.* [6] explicitly showed that it is indeed the results of finite precisions of numerical analysis and claimed that synchronization never occurs statistically with infinite precision calculations. On the other hand, several studies [7] reported the importance of the profiles of the applied common noises, especially a negative bias of the noise distributions, in synchronization by common noises. The most recent, Lai and Zhou [8], showed the possibility that synchronization can be achieved with symmetric common noises. The connection between these phenomena and the chaotic transition of random dynamical systems has been mentioned in several articles. There is, however, no explicit analysis of this connection yet, as far as we know.

In this Letter, we analyze the mechanism behind the connection between the chaotic transitions of random maps and synchronization of chaotic maps by external common noises. The main motivation of this study comes from the speculation that the origin of the extreme intermittent behavior, which was observed by YOC, is *on-off intermittency.* We investigate the laminar distributions at the onset to prove our speculation. We also closely analyze that synchronization of coupled chaotic maps can be actually achieved by applying common noises of a certain profile by explicitly obtaining the synchronization parameter region for coupled logistic maps with common noises.

As we mentioned above, YOC considered the motion of floating particles on a surface of incompressible fluid. As a result they obtained the following $2N$-dimensional random map for N-floating particles [1,9] (it should be noted that the following map can be regarded as N coupled chaotic maps by common noises):

$$\begin{aligned} x_{n+1}^i &= x_n^i + \alpha^{-1}(1 - e^{-\alpha})y_n^i \qquad \mathrm{mod}(2\pi), \\ y_{n+1}^i &= e^{-\alpha}y_n^i + 0.5\sin(x_{n+1}^i + \xi_n), \end{aligned} \tag{1}$$

where $i = 1, \ldots, N$ and ξ is a uniform random noise with width 2π, which means ξ takes on a number between 0 and 2π with equal probability. In our study, we take the width of common random noises as $2\pi b$ and vary b from

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1 to 2. In this way, we are able to modify the profiles of common noises, so our noises are not uniform in effect. The numerical study shows that the system spreads over the snapshot attractor for small α and shrinks to a point for larger α after enough iterations. Since the transition value of α should be independent of the number of particles of the system, it is enough to investigate a two-particle system to determine the transition point.

When $b = 1$, we numerically find that the transition point of the parameter α is very close to $\alpha = 0.27$. At this parameter value, however, it takes a very long time (more than 10^7 iterations) to get the synchronization (coalescence of two particles). And if we add very small independent noise to each particle, the system is unable to maintain the synchronization. So the synchronization is not actual but accidental due to finite precisions of numerical analysis. Now when we change the noise profile by varying b, the actual complete synchronization is achieved at $\alpha = 0.27$ for $1.258 < b < 1.415$. From this observation, we can see that the profiles of common noises *do* have an impact on the synchronization process of chaotic maps and that synchronization parameter regions can be obtained for the set of parameters (α, b). At the boundary of the synchronization parameter region, we can expect extreme intermittent fluctuations of the attractor size. Figure 1 plots time series of dispersions of y variables, i.e., $S^N = \{N^{-1}\sum(y^i - \overline{y})^2\}^{1/2}$ which could be interpreted as the vertical size of the snapshot attractor, for $\alpha = 0.27$ and $b = 1.415$, i.e., at the boundary of the synchronization parameter region, where $\overline{y} = N^{-1}\sum y^i$ is the average vertical position of the snapshot attractor. All four plots exhibit extreme intermittent fluctuations of their vertical size.

In order to uncover the origin of this intermittent behavior, we numerically investigate the scaling of the normalized laminar distributions near and away from the transition boundary, i.e., $(\alpha, b) = (0.27, 1.415)$, for several different particle numbers. The normalized laminar distributions against the laminar length for 2-particle and 500-particle cases are plotted in Fig. 2. For $\alpha = 0.27$, i.e., at the onset, both figures agree well with the known $-\frac{3}{2}$ scaling of on-off intermittency. Notice that the number of long laminars decreases rapidly as α decreases. Carrying out all of the above numerical calculations, we add an extremely small (an order of 10^{-18}) independent uniform noise to each particle at every iterated step to avoid accidental synchronization [5].

So far we have shown that the vertical size of the snapshot attractor temporally exhibits on-off intermittency near the boundary of the synchronization parameter region. Our claim is that this is the characteristic behavior of the transition to chaos of random maps.

In order to show our argument more explicitly we analyze a simpler system [10], which is two coupled logistic maps with common random noises.

$$\begin{aligned} x_{n+1} &= \lambda x_n(1 - x_n) + \xi_n \qquad \mathrm{mod}(1), \\ x'_{n+1} &= \lambda x'_n(1 - x'_n) + \xi_n \qquad \mathrm{mod}(1), \end{aligned} \tag{2}$$

where $\lambda = 4.0$ and ξ is a uniform common noise with width b and bias a, which means noises take on a number between a and $a + b$ with equal probability. This is the same system as MB originally studied except that we take modular for each equation at each iteration step to prevent the trajectory from diverging and to avoid an awkward selection process of random noises. It is well known that the maximal Lyapunov exponent for this system is positive

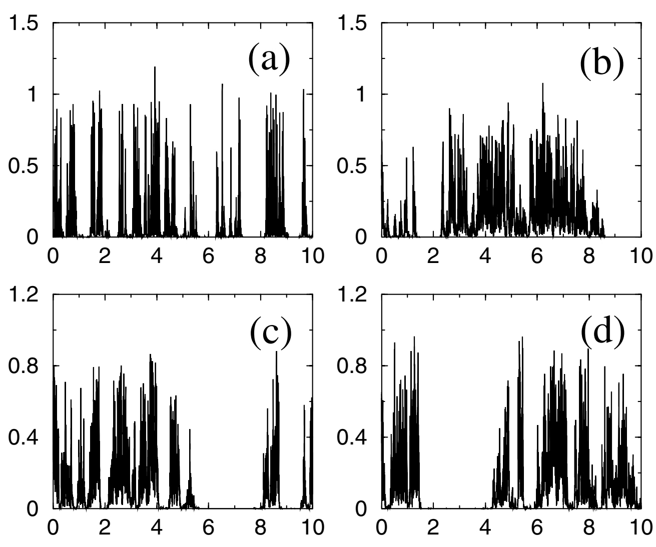

FIG. 1. Plots of root-mean-squared vertical size of the snapshot attractor S_N vs iteration number $n(\times 10^3)$ for $\alpha = 0.27$ and $b = 1.415$ show extreme intermittent fluctuations. The numbers of particles are (a) 2, (b) 500, (c) 5×10^4, and (d) 10^6, respectively, in the order of reading text.

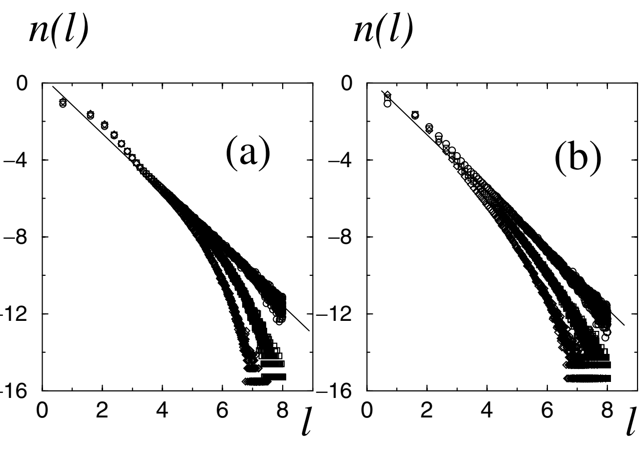

FIG. 2. The normalized laminar distribution $n(l)$ vs laminar length l in log-log scale for (a) $N = 2$ and (b) $N = 500$, respectively. The results for three different α's, i.e., $\alpha = 0.15$ (circle), $\alpha = 0.20$ (square), and $\alpha = 0.27$ (diamond) are plotted, respectively, in (a). The results for three different α's, i.e., $\alpha = 0.20$ (circle), $\alpha = 0.23$ (square), and $\alpha = 0.27$ (diamond) are plotted, respectively, in (b), where $b = 1.415$ for all the cases. The solid line represents the slope $-\frac{3}{2}$ line. At the onset, i.e., $\alpha = 0.27$ and $b = 1.415$, the slopes agree very well with the characteristic scaling exponent of on-off intermittency.

when $\lambda = 4.0$. If we regard ξ as a constant parameter, we can plot the bifurcation diagram for one of Eq. (2) as ξ varies. From Fig. 3(a) we can intuitively see that the relative motion of two particles can attract each other on the average if ξ varies in the simple periodic region of the diagram. In this case, the transversal Lyapunov exponent should be negative. If ξ varies through from the more complex periodic to the chaotic region, the system can be chaotic, so that the transversal Lyapunov exponent could become positive. In between, there must be a continuous set of parameter values (a, b) which divides the synchronization region from the desynchronization region in the parameter space. This continuous set of parameter values can be determined by the condition that the transversal Lyapunov exponent is zero at the boundary.

To calculate the transversal Lyapunov exponent ν of the system near the transition point we use the error dynamics, which is defined as the dynamics of difference between x and x'. Since the error dynamics of Eq. (2) does not contain any information about common noises ξ, we transform the above equation as follows [11]:

$$y_n = x_n - \xi_n . \tag{3}$$

Then, in the y representation, the original equations become

$$\begin{aligned} y_{n+1} &= \lambda(y_n + \xi_n)(1 - y_n - \xi_n), \\ y'_{n+1} &= \lambda(y'_n + \xi_n)(1 - y'_n - \xi_n). \end{aligned} \tag{4}$$

Now, we can obtain nontrivial error dynamics which describes the relative motion of two particles (or two coupled equations) as follows:

$$z_{n+1} = y_{n+1} - y'_{n+1} \simeq a_n z_n + O(z_n^2), \tag{5}$$

where $a_n = \lambda(1 - 2y_n - 2\xi_n)$ as a dynamic parameter. Near the transition point, the relative motion z is small for most of the time, so higher order terms of z_n can be neglected. From the last line of Eq. (5), we can immediately see that the error dynamics of the system is of the Heagy, Platt, and Hammer's on-off form [3,12]. From the bifurcation diagram [Fig. 3(b)], we can expect to observe on-off intermittency when the dynamic parameter a_n varies through the bifurcation point. Depending on the varying range of the dynamic parameter a_n, the system exhibits different dynamical behaviors. The transversal Lyapunov exponent ν can be estimated by using the following relation:

$$\nu = \int d\xi \int dy \ln\{\lambda(1 - 2y - 2\xi)\} P(y, \xi), \tag{6}$$

where $P(y, \xi)$ is a normalized probability distribution.

Using the above equation, we are able to numerically determine the boundaries of the parameters (a, b) where the Lyapunov exponent ν changes its sign. Figure 4 shows the parameters (a, b) region where two particles can make complete synchronization. In order to make sure that the synchronization actually occurs in the region, finite accuracy calculations [6] are done for two different sets of parameter values near the boundary point P in Fig. 4. Table I is the result.

It is interesting to note that the noise parameter values MB used (solid line in Fig. 4) are far from the actual synchronization region. As a matter of fact, with MB's original scheme, it is actually impossible to get the noise profile which leads the system to synchronize. This shows that the profile of noise distribution, with which the system synchronizes, critically depends on the system under consideration. So it is not surprising to devise some systems in which the uniform symmetric noise can lead them to synchronize [8].

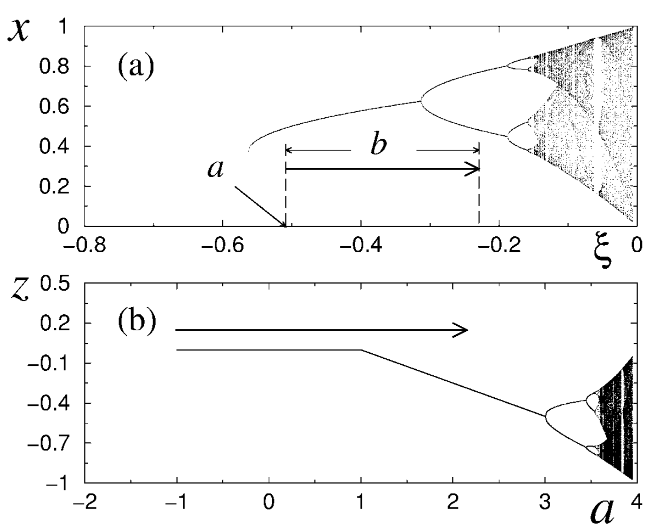

FIG. 3. Bifurcation diagram for (a) Eq. (2) and (b) Eq. (5) as ξ and $a = \lambda(1 - 2y - 2\xi)$ vary when $\lambda = 4.0$. Depending on the noise parameters, i.e., bias a and width b, two coupled logistic maps could exhibit different characteristic behaviors.

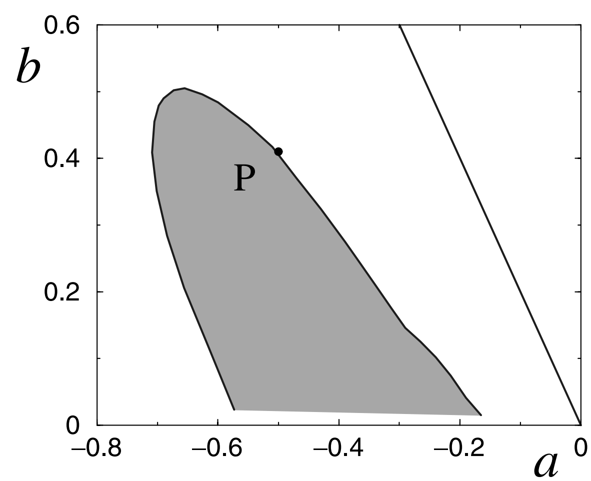

FIG. 4. The shaded region is the synchronization region of noise parameters (a, b) where the transversal Lyapunov exponent ν is negative. Near the boundary of this region which is calculated from Eq. (6), the system exhibits on-off intermittency. The solid line represents the possible parameter values of MB's original study.

TABLE I. Precisions (in digits) against the average number of iterations which are needed to observe synchronization for 100 different sets of initial values. The inside point is $a = -0.52$, $b = 0.42$, the outside point is $a = -0.49$, $b = 0.42$, and the boundary point is $a = -0.51$, $b = 0.42$. At the inside point, the needed iterations grow *linearly* as precision increases. This means the average distance between the trajectories converges exponentially to zero. On the contrary, at the outside point, the needed iterations grow *exponentially* as precision increases. In this case, the observed synchronization is a false one caused by the finite precision calculation.

Precision	5	10	15	20	25	30
Inside	61	181	346	549	728	839
Outside	96	602	2382	8081	21 077	50 529

The same argument can be easily extended to the N-particle case. Now, consider S_{n+1}^N which is the root-mean-squared distance from the average location of the particles, i.e., $\overline{y}$.

$$S_n^N = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(y_n^i - \overline{y}_n)^2}\,, \tag{7}$$

where $\overline{y}_n = \frac{1}{N}\sum_{i=1}^{N} y_n^i$. Since $|\partial y_{n+1}/\partial y_n| = \lambda(1 - 2\xi_n - 2y_n) = a_n(\xi_n, \overline{y}_n)$,

$$S_{n+1}^N \simeq \left|\frac{\partial y_{n+1}}{\partial y_n}\right|_{y_n=\overline{y}_n} S_n^N = a_n(\xi_n, \overline{y}_n) S_n^N\,. \tag{8}$$

So this is formally the same as Eq. (5).

Figure 5 shows that at the point P on the boundary of the synchronization region (see Fig. 4), the scaling of the normalized distribution of laminar length l is in good agreement with the scaling of on-off intermittency. The observed shoulder of the laminar distributions comes from the small [12] independent noise (an order of 10^{-14}) which we added at each step to avoid an accidental synchronization.

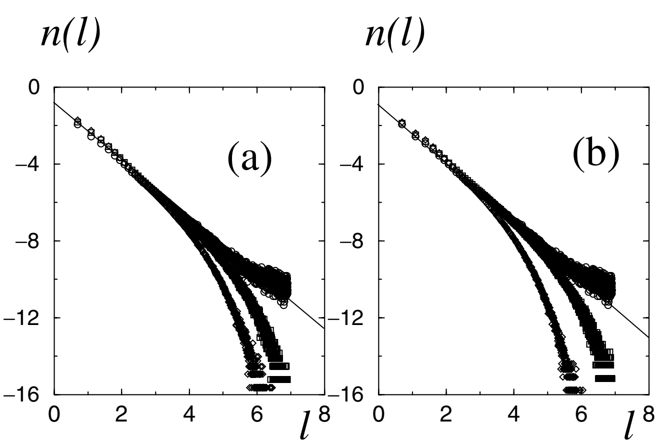

FIG. 5. At the point P on the boundary of the synchronization region (Fig. 4), the normalized laminar distribution $n(l)$ is plotted against the laminar length l in log-log scale. (a) 2-particle case, $a = -0.5$ (circle), -0.47 (square), and -0.44 (diamond). (b) 500-particle case, $a = -0.5$ (circle), -0.47 (square), and -0.45 (diamond). The widths of noises b are fixed at 0.41 in both figures and the solid line represents the slope $-\frac{3}{2}$ line.

In conclusion, we have studied the relation between chaotic transitions of random dynamical systems and chaotic maps coupled with external common noises. Common noises do have an impact on synchronization if we change the profile of noises appropriately even though the system has a positive maximal Lyapunov exponent. Near the transition point, the random dynamical systems exhibit intermittent behavior. By studying the scaling of laminar distributions, we have shown that the origin of the extreme intermittent behavior which YOC observed is on-off intermittency. We think that this type of behavior is an intrinsic characteristic of the transition to chaos of random dynamical systems. All these observations lead us to conclude that *chaotic transitions through on-off intermittency could be one of the routes to chaos of random dynamical systems* as period doubling, intermittency, quasiperiodicity, and crises are those of deterministic dynamical systems. This might be the answer that YOC looked for ten years ago.

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[1] L. Yu, E. Ott, and Q. Chen, Phys. Rev. Lett. **65**, 2935 (1990).
[2] N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); P. W. Hammer, N. Platt, S. M. Hammel, J. F. Heagy, and B. D. Lee, Phys. Rev. Lett. **73**, 1095 (1994).
[3] J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994).
[4] A. Maritan and J. Banavar, Phys. Rev. Lett. **72**, 1451 (1994).
[5] A. S. Pikovsky, Phys. Rev. Lett. **73**, 2931 (1994); A. Maritan and J. Banavar, Phys. Rev. Lett. **73**, 2932 (1994).
[6] L. Longa, E. M. F. Curado, and F. A. Oliveira, Phys. Rev. E **54**, R2201 (1996).
[7] R. Brown, N. F. Rulkov, and N. B. Tufillaro, Phys. Lett. A **196**, 201 (1994); H. Herzel and J. Freund, Phys. Rev. E **52**, 3238 (1995); G. Malescio, Phys. Rev. E **53**, 6551 (1996); P. M. Gade and C. Basu, Phys. Lett. A **217**, 21 (1996); E. Sanchez, M. A. Matias, and V. Perez-Munuzuri, Phys. Rev. E **56**, 4068 (1997).
[8] C. H. Lai and C. Zhou, Europhys. Lett. **43**, 376 (1998).
[9] G. M. Zaslavsky, Phys. Lett. **69A**, 145 (1978).
[10] The following argument is applied to generic random dynamical systems as far as the systems are synchronized by appropriate common noises.
[11] Two systems [Eq. (2) and Eq. (4)] describe the same dynamics since the transformation [Eq. (3)] is a *homeomorphism*.
[12] N. Platt, S. M. Hammel, and J. F. Heagy, Phys. Rev. Lett. **72**, 3498 (1994).