Creation of Localized Optical Waves that Do Not Obey the Radiation Condition at Infinity

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(Received 27 December 1999)

We show that the diffraction of a shocking optical pulse formed in a nonlinear transparent dielectric creates an optical missile, or localized radiation field whose amplitude and energy decays are slower than $1/R$ and $1/R^2$, respectively, far from the aperture. Dispersion does not eliminate, but limits missile behavior to a finite range. Experimental techniques for optical missile generation are suggested.

PACS numbers: 42.65.Re, 42.65.Tg

The rapid advances of laser technology in the past decade have made the production of intense pulses of light with only a few optical oscillations feasible. Because of the high intensities concentrated in these pulses, of the order of tens of TW/cm^2, the possibility has been recently [1,2] proposed, as first conjectured by Rosen [3], of nonlinearly generating optical shocks; that is, steepening and breaking the optical cycles upon propagation of an intense femtosecond pulse in a nonlinear transparent dielectric. These shocks can form over a propagation distance of a few micrometers, prior to the formation of the more known shocks in the pulse envelope [1]. Optical cycle steepening in realistic nonlinear dielectrics (fused silica), including dispersive and absorption effects of any order, has been recently evidenced from theoretical and numerical studies from Maxwell equations [1,2].

From the early work by Christov [4], on the other hand, it is known that the diffraction properties of few-cycle pulses may differ substantially from those of quasimonochromatic light. Diffraction is a dispersive phenomenon in the sense that different frequencies diffract differently. The diffraction of femtosecond pulses, having a broad content of frequencies, will then induce some dispersionlike transformations in the pulse form—transformations which need to be understood and therefore are being intensively studied [5–7]. The best understood among them is perhaps the differentiation of the pulse temporal form upon propagation from an aperture to the far field, or time-derivative effect [5,8]. Closely connected with it [8] is the concept of "electromagnetic missile," described for the first time by Wu [9] in the context of antenna theory. Electromagnetic missiles are radiation fields from spatially localized sources which do not obey the radiation condition at infinity ($1/R^2$ decay in energy), but decay at slower rates along a specified direction away from the source [9]. They can be generated, in principle, by driving the localized source with pulses having a singular derivative stronger than $\sqrt{t}$, i.e., t^μ, with $\mu < 1/2$ [10], but their practical realization has run up against the difficulty of physically producing pulses with the required sudden jumps [11].

In this Letter, we show that an optical shock formed in a nonlinear transparent dielectric contains the required jumps to produce the missile effect. The diffraction of a shock pulse by an aperture originates a radiation field whose amplitude and energy present a far field decay significantly slower than $1/z$ and $1/z^2$, respectively, along the perpendicular direction z to the aperture. Numerical simulations including dispersion and absorption indicate that the missile effect survives up to a distance proportional to the steepest gradient of the cycles reached in the nonlinear dielectric. These optical missiles could be of interest for many practical applications, as distance measurements, alignment, as well as transmission of information and energy over long distances.

Let us first derive our basic equation for femtosecond pulse propagation. We consider a polarized pulsed beam of light $E(\mathbf{x}_\perp, z, t)$, $\mathbf{x}_\perp \equiv (x, y)$, propagating along the positive z direction according to the wave equation in a material medium, $\Delta E - c^{-2}\partial_{tt}E = \mu_0 \partial_{tt} P$, where P is the medium polarization. As shock formation is expected to occur under weak dispersion conditions [1–3], we neglect, for the moment, material dispersion by writing the polarization as $P = \varepsilon_0 \chi^{(1)} E + P_{\rm nl}(E)$, where $P_{\rm nl}$ is a nonlinear function of E. The linear part of P can be embedded in the second term of the left-hand side of the wave equation by identifying c with the light velocity in the medium. Next, by introducing the local coordinates $t' = t - z/c$, $z' = z$, we extract from E its rapid variation with z owing to the pulse transport at c; then the remainder dependence of $E(\mathbf{x}_\perp, z', t')$ on the new propagation coordinate z' describes only pulse changes due to diffraction and nonlinearity. If, moreover, these changes are slow enough so that $|\partial_{z'}E| \ll |\partial_{t'}E|/c$, the following first order propagation equation can be readily derived from the wave equation:

$$(2/c)\partial_{z't'}E = \Delta_\perp E - \mu_0 \partial_{t't'} P_{\rm nl}, \qquad (1)$$

where $\Delta_\perp$ is the transversal Laplace operator. Note that by writing E and $P_{\rm nl}$ as enveloped carrier oscillations, Eq. (1) transforms into the nonlinear envelope equation [12] (in nondispersive media) under the slowly evolving wave approximation (SEWA). Thus Eq. (1) describes wave propagation within the same degree of approximation—E does not significantly change due to diffraction and nonlinearities as it covers its own characteristic axial length of variation—but for an arbitrary wave form E, including those with subcycle structure. The present extension of the SEWA allows us to perform the analysis of shock formation and diffraction on a unified basis, which, moreover, will accurately reproduce the predictions of Maxwell equations related to these phenomena.

In the case of a plane pulse ($\Delta_\perp E = 0$) in a nonlinear material, Eq. (1) reduces to the quasilinear equation $\partial_{z'}E + \sigma(E)\partial_{t'}E = 0$, where $\sigma(E) = (c/2) \times \mu_0(dP_{\rm nl}/dE)$, which describes the formation of a shock [13] from a smooth pulse $E(0,t') = f(t')$, as illustrated in Figs. 1(a) and 1(b) for a medium with cubic nonlinearity $P_{\rm nl} = \varepsilon_0\chi^{(3)}E^3$. Indeed the solution $E(z,t')$ of the quasilinear equation can be obtained [13] in parametric form by the method of characteristics,

$$E = f(\tau), \qquad t' = \tau + \sigma(\tau)z\,, \tag{2}$$

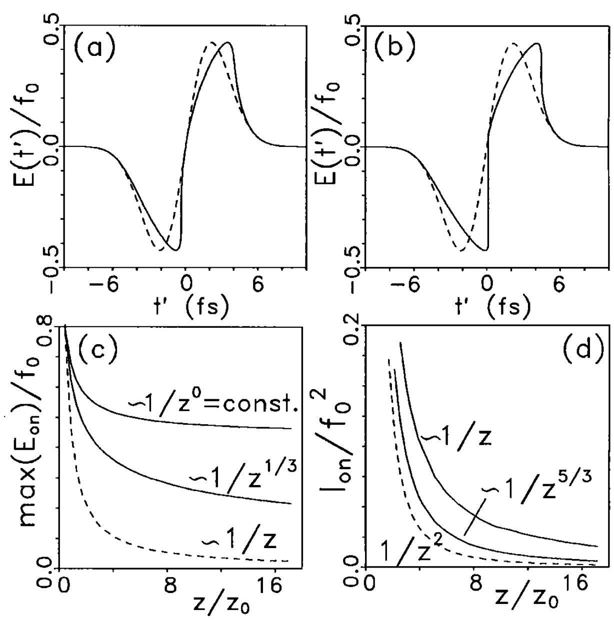

FIG. 1. Propagation in a Kerr medium ($P_{\rm nl} = \varepsilon_0\chi^{(3)}E^3$) of the input "one-cycle" pulse $f(t') = (t'/T)f_0\exp(-t'^2/T^2)$, $T = 3$ fs, $f_0^2\chi^{(3)} = 0.01$ (dashed line). (a) At $z' = z'_{\rm sh,1} = 0.145$ mm a first shock is in imminent formation at $t'_{\rm sh,1} = -0.357$ fs. (b) At $z' = z'_{\rm sh,2} = 0.215$ mm a second shock is being formed at $t'_{\rm sh,2} = 4.454$ fs, and the pulse is discontinuous in the first shock. (c) On-axis amplitude decay of the pulses in (a) and (b) on propagation from an aperture of radius $r = 0.5$ mm. (d) On-axis energy decay of the pulses in (a) and (b) on propagation from the aperture.

where $\sigma(\tau) = \sigma[f(\tau)]$. The shock in the electric field [1–3] is characterized by the condition $\partial_{t'}E = \infty$. From Eq. (2) we obtain $\partial_{t'}E = f_\tau(\tau)/[1 + \sigma_\tau(\tau)z]$ (the subscript τ denotes differentiation with respect to τ), which diverges at $z = -1/\sigma_\tau(\tau)$. The shock then appears for the first time at the distance $z_s = \min[-1/\sigma_\tau(\tau)]$, with $z_s > 0$. Thus setting $\sigma_{\tau\tau}(\tau) = 0$, one determines a certain value of τ, say τ_s, which yields a shock distance $z_s = -1/\sigma_\tau(\tau_s)$. At z_s, the shock takes place at the local time $t'_s = \tau_s - \sigma(\tau_s)/\sigma_\tau(\tau_s)$.

After recalling these properties, let us now investigate how the slope of E goes to infinity when t' approaches the shock time t'_s at the distance z_s of imminent wave breaking, as in Fig. 1(a). For this, we let $(\tau - \tau_s) \to 0$; then $(t' - t'_s) \to 0$ also, and the ways both approach zero are related by

$$(t' - t'_s) \simeq -(\tau - \tau_s)^3\sigma_{\tau\tau\tau}(\tau_s)/6\sigma_\tau(\tau_s)\,. \tag{3}$$

To obtain Eq. (3) we have written $(t' - t'_s) = (\tau - \tau_s) + [\sigma(\tau) - \sigma(\tau_s)]z_s$, introduced $z_s = -1/\sigma_\tau(\tau_s)$, the power expansion $\sigma(\tau) = \sigma(\tau_s) + \sigma_\tau(\tau_s)(\tau - \tau_s) + \ldots$, used that $\sigma_{\tau\tau}(\tau_s) = 0$, and retained the first nonvanishing term. For particular forms of $P_{\rm nl}$ and $f(t')$, one can get $\sigma_{\tau\tau\tau}(\tau_s) = 0$. In this case the right-hand side of Eq. (3) will be $\propto(\tau - \tau_s)^4$, and so on. Finally, E in the vicinity of τ_s can be approached by $E \simeq f(\tau_s) + f_\tau(\tau_s)(\tau - \tau_s)$, or from Eq. (3),

$$E(z_s,t') \simeq f(\tau_s) - f_\tau(\tau_s)\left[\frac{6\sigma_\tau(\tau_s)}{\sigma_{\tau\tau\tau}(\tau_s)}(t' - t'_s)\right]^{1/3}. \tag{4}$$

This equation gives the explicit temporal behavior of the electric field around the shock time t'_s when the wave is going to break (z_s), and shows that the slope of E approaches infinite as that of $(t' - t'_s)^{1/3} = \mathrm{sgn}(t' - t'_s)|t' - t'_s|^{1/3}$ [in the case $\sigma_{\tau\tau\tau}(\tau_s) = 0$, the corresponding functional form would be $\mathrm{sgn}(t' - t'_s)|t' - t'_s|^{1/4}$].

At $z > z_s$, E jumps from one to another branch of the three-valued expression (2) for E [Fig. 1(b)]. The condition for such a discontinuous solution of the quasilinear equation to represent physically admissible wave propagation is the contents of the Rankine-Hugoniot theorem [13], $(E_i - E_f)(dt'_s/dz') = (c\mu_0/2)(P^i_{\rm nl} - P^f_{\rm nl})$, relating the "velocity" of propagation $1/v_s \equiv dt'_s(z)/dz$ of the discontinuity $t'_s(z)$ with the jumps $E_i - E_f$ and $P^i_{\rm nl} - P^f_{\rm nl}$ in E and $P_{\rm nl}$. This velocity reproduces Rosen's result for cubic polarization from the Maxwell equations [3] when $\chi^{(3)}E^2 \ll 1$, as expected from the assumptions of the SEWA.

We can then conclude that an intense enough optical pulse in an ideal nonlinear dielectric develops an infinite gradient of functional form $\mathrm{sgn}(t')|t'|^\mu$, with $0 \le \mu \le \frac{1}{3}$ (the case $\mu = 0$ describing wave breaking).

On the other hand, we know from the mathematical theory of electromagnetic and acoustic missiles that the above infinite gradients are within the required ones [9,10] for the signal driving a localized source to produce the missile effect. Specifically, missile behavior is obtained for $0 \le \mu < \frac{1}{2}$. From its first description by Wu [9], however, published theoretical and experimental [14] methods of missile generation put emphasis on finding spatial dispositions of electric currents, radiating point, or line charges, to conform with the plane wave fronts of the launched pulse; but the difficult problem of how to produce the required infinite gradients is relegated to a less important emphasis, or left aside entirely. Analytical pulse forms as $t^\mu \exp(-\alpha t)$, $0 \le \mu < \frac{1}{2}$ [10], or rectangle functions [11] are often used, at the same time that our present inability to physically generate them is recognized [11]. Similarly, Ffowcs [15] studied the missile effect for the diffraction of acoustic pulses, modeled as Heaviside step functions. Moreover, he suggested that nearly sharp front pulses can be generated by the impact of a vortical flow on an infinitely thin supersonic aerofoil. The present analysis suggests that pulse shocking in a nonlinear dielectric is a suitable physical mechanism to generate the needed rise time in an optical signal. Driving a finite source, e.g., a circular aperture, with the shock pulse, a missile in the optical range of frequencies can be created.

Let us then analyze its asymptotic amplitude and energy decays. The free propagation of a pulse $f(t')$ after diffraction by a circular aperture of radius r_0 at $z = 0$ can be described by Eq. (1) with $P_{\rm nl} = 0$, i.e., $\partial_{z't'}E = (c/2)\Delta_\perp E$, which in the frequency domain is the well-known paraxial wave equation $\partial \hat{E}/\partial z' = (c/2i\omega)\Delta_\perp \hat{E}$. The propagation of each component frequency ω, of amplitude $\hat{f}(\omega)$, is then described by the Huygens-Fresnel integral for a circular aperture, and can be found in standard textbooks. In particular, the on-axis ($\mathbf{x}_\perp = 0, z > 0$) field is $\hat{E}_{\rm on}(z,\omega) = \hat{f}(\omega)[1 - \exp(-i\omega r_0^2/2cz)]$, which in time domain yields

$$E_{\rm on}(z,t') = f(t') - f(t' - r_0^2/2cz). \qquad (5)$$

A similar expression was found from Maxwell equations for the electric field from a disk of surface current $f(t')$ [11]. For z sizably greater than the diffraction length $z_0 \equiv r_0^2/2cT$, where T is a typical rise time of $f(t')$, Eq. (5) can be approached by $E_{\rm on}(z) \simeq f_{t'}(t')(r_0^2/2cz)$. This is the time-derivative effect, valid for very general diffracting apertures [8], along with the classical amplitude decay $1/z$ in the far field. If the pulse contains a local rise time $T = 0$, however, the $1/z$ regime is never reached. According to Eq. (5), the on-axis field from a diffracted discontinuous pulse ($\mu = 0$) does not decay at the time of discontinuity, as shown in Fig. 2. For a general shock pulse $f(t') \sim \mathrm{sgn}(t')|t'|^\mu$, $0 \le \mu \le \frac{1}{3}$, Eq. (5) leads to a field decay $1/z^\mu$ at the shock time. In Fig. 1(c) we show the nonclassical amplitude decays (solid lines) from diffracted shock pulses in comparison with the classical decay $1/z$ (dashed line) for the "same" pulse prior passage through the nonlinear medium (i.e., with the same maximum amplitude and energy, but without shock).

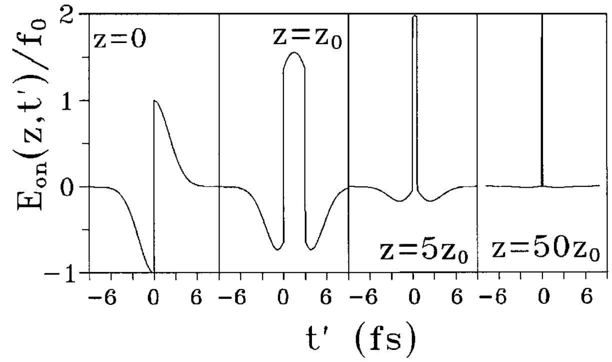

FIG. 2. On-axis pulse form after diffraction from a circular aperture of radius $r_0 = 1.0$ mm. The pulse at the aperture is $f(t') = \mathrm{sgn}(t')\exp(-t'^2/T^2)$, $T = 3$ fs.

From Eq. (5) the on-axis time integrated intensity $I_{\rm on} = \int E_{\rm on}^2\, dt$ can be found to be

$$I_{\rm on}(z) = \int_0^\infty d\omega\, |\hat{f}(\omega)|^2[1 - \cos(\omega r_0^2/2cz)] \sim \int_{2cz/r_0^2}^\infty d\omega\, |\hat{f}(\omega)|^2, \qquad (6)$$

the last expression giving the asymptotic behavior for large z [9]. Introducing the asymptotic form of the spectra of shock pulses [16] $\hat{f}(\omega) \sim 1/\omega^{1+\mu}$, $0 \le \mu \le \frac{1}{3}$, we obtain the energy density asymptotic decay $I_{\rm on} \sim 1/z^{1+2\mu}$. Figure 1(d) shows these nonclassical energy decays for diffracted shock pulses in comparison with the decay $1/z^2$ for the nonshocked pulse.

Before concluding, it remains to analyze how the inclusion of material dispersion may alter the preceding results. Recent numerical studies from the Maxwell equations [1,2] indicate that dispersion does not inhibit the process of shock formation, i.e., of cycle steepening, but prevents the field from breaking, and can limit the steepest gradient reached. Figure 3(a) is taken from Ref. [2], and shows incomplete carrier-shock formation (solid line) in the incoming pulse $E(t') = f_0\,\mathrm{sech}(t/t_0)\cos(\omega_0 t')$ (dashed line) on propagation in fused silica, modeled as a triple-resonant Lorentz medium with Kerr nonlinearity. Dispersion of any order is then included, and in particular, a group velocity dispersion of 2.8 fs^2/mm at the selected frequency $\omega_0 = 1.52$ fs^{-1}. From our Eq. (5), we have calculated the diffracted on-axis pulse form. A significant survival of the peak amplitude with respect to the diffracted nonshocked pulse is apparent [Fig. 3(b)], even when an infinite slope was not reached in the material. The peak amplitude versus z is depicted in Fig. 3(c), and shows an increase of the depth of field ($1/e$ amplitude decay) of nearly three times. In Fig. 3(d) we have multiplied the curves of Fig. 3(c) by z, to show that the decay of the diffracted shocking pulse is slower than $1/z$ over three or four times the distance z_0 of the nonshocked pulse. Additional numerical simulations with shocks of different thicknesses or rise times T allow us to estimate the range of

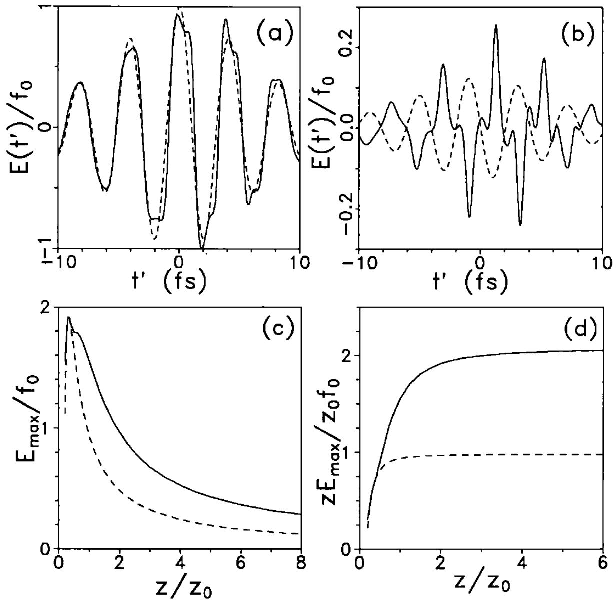

FIG. 3. In all figures the dashed lines refer to the pulse $f(t') = f_0\,\mathrm{sech}(t/t_0)\cos\omega_0 t'$, $t_0 = 5$ fs, $\omega_0 = 1.52\ \mathrm{fs}^{-1}$, and the solid lines to its temporal form after a propagation distance 7.5 μm in fused silica, with $f_0^2\chi^{(3)} = 0.06$. They are shown in (a). (b) On-axis pulse forms after a propagation distance $z = 9z_0$ behind an aperture of radius $r_0 = 0.5$ mm. The quantity $z_0 = r_0^2/2cT$ is the diffraction length, with $T = 1/\omega_0$ the characteristic rise time of the nonshocked pulse. (c) On-axis peak amplitude, and (d) on-axis peak amplitude multiplied by z/z_0 versus propagation distance from the aperture.

missile behavior by $r_0^2/2cT$, where the rise time is defined as $T = \max|f|/\max|f_{t'}|$ (giving the reasonable values $T = 1/\omega_0$, or phase of 1 radian for a sinusoidal wave, and $T = 0$ for an actual shock). To minimize the shock rise time, the selected dielectric should have a broad spectral band of small dispersion, high 3 or 5 order nonlinear susceptibility (organic materials as PTS with $\chi^{(3)}$ 6 orders of magnitude higher than fused silica are today available [17]) and high damage threshold to allow for high pulse intensity.

To sum up, we have found that nonlinear optical carrier shocks contain the required sharp gradients to create optical missiles. Slow decays between $1/z^0$ and $1/z^{1/3}$ in amplitude, and between $1/z$ and $1/z^{5/3}$ in energy can be reached in a transparent dielectric. In practice, we have shown that shocking a pulse in a suitable nonlinear medium prior to diffraction results in a substantial improvement of the beam depth of field. This effect could be experimentally verified by tightly focusing a pulse (to achieve the needed intensities), and placing in the focal region (where the pulse can be regarded as plane) a thin nonlinear dielectric having an opaque outer face with a hole of small radius r_0. Owing to the smallness of the aperture, however, this setup yields a small depth of field (with or without missile effect). To be suitable for large distance applications, we must let the shock pulse expand behind the thin dielectric, and then collimate its expanding spherical wave front, e.g., with a parabolic reflective dish (whose finite transversal radius r_0 is now the effective aperture) placed at the appropriate distance behind the focus. A similar arrangement was used in Ref. [14] to investigate the missile effect from a point antenna placed at the dish focus. Note that placing the thin dielectric in place of the point source, the reflected pulse at the far field will resemble the second derivative of the shock pulse, a fact which can enhance the shock gradient, and therefore lessen the smoothing effects of material dispersion.

The authors thank the Comisión Interministerial de Ciencia y Tecnología of Spain for partial support (PB 95-0426), and L. Gilles for help in numerical simulations.

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