Particles Sliding on a Fluctuating Surface: Phase Separation and Power Laws

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We study a system of hard-core particles sliding locally downwards on a fluctuating one-dimensional surface characterized by a dynamical exponent z and no overall tilt. In numerical simulations, an initially random particle density is found to coarsen and obey scaling with a growing length scale $\sim t^{1/z}$. The structure factor deviates from the Porod law for the models studied. The steady state is unusual in that the density-segregation order parameter shows strong fluctuations. The two-point correlation function has a scaling form with a cusp at small argument which we relate to a power law distribution of particle cluster sizes. Exact results on a related model of surface depths provide insight into this behavior.

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How do density fluctuations evolve in a system of particles moving on a fluctuating surface? Can the combination of random vibrations and an external force such as gravity drive the system towards a state with large-scale clustering of particles? Such large-scale clustering driven by a fluctuating potential represents an especially interesting possibility for the behavior of two coupled systems, one of which evolves autonomously but influences the dynamics of the other. Semiautonomous systems are currently of interest in diverse contexts, for instance, advection of a passive scalar by a fluid [1], phase ordering in rough films [2], the motion of stuck and flowing grains in a sandpile [3], and the threshold of an instability in a sedimenting colloidal crystal [4].

In this paper, we show that there is an unusual sort of phase ordering in a simple model of this type, namely, a system of particles sliding locally downwards under a gravitational field on a fluctuating one-dimensional surface with zero global average slope. The surface evolves through its own dynamics, while the motion of particles is guided by local downward slopes; since random surface vibrations cause slope changes, they constitute a source of nonequilibrium noise for the particle system. The mechanism which promotes clustering is simple: fluctuations lead particles into potential minima or valleys, and once together the particles tend to stay together, as illustrated in Fig. 1. The question is whether this tendency towards clustering persists up to macroscopic scales. We show below that in fact the particle density exhibits coarsening towards a phase-ordered state. This state has uncommonly large fluctuations which affect its properties in a qualitative way, and makes it quite different from that in other driven, conserved systems which exhibit coarsening [5].

It is useful to state our principal results at the outset: (i) In an infinite system, an initially randomly distributed particle density exhibits coarsening with a characteristic growing length scale $\mathcal{L}(t) \sim t^{1/z}$, where z is the dynamical exponent governing fluctuations of the surface. For the models we study, the scaled structure factor varies as $|k\mathcal{L}(t)|^{-(1+\alpha)}$ with $\alpha < 1$, which represents a marked deviation from the Porod law ($\alpha = 1$) for coarsening systems [6]. Further, a finite system of size L reaches a steady state with the following characteristics: (ii) The magnitude of the density-segregation order parameter has a nonzero time-averaged value, but shows strong fluctuations which do not decrease as L increases. (iii) The static two-point correlation function $C(r)$ has a cusp at small values of the scaled separation $|r/L|$. (iv) The sizes l of particle clusters are distributed according to a power law for large l, up to an L-dependent cutoff. These results are established by extensive numerical simulations. Further, the properties (iii) and (iv) are shown to be related, using the independent interval approximation [7] applied to the cluster distribution. Also, we define a related coarse-grained depth model of the surface, and show analytically that the steady state characteristics (ii)–(iv) hold for this model.

The sliding particle (SP) model is defined as a lattice model of particles moving on a fluctuating surface. Both the particles and the surface degrees of freedom are represented by ± 1 valued Ising variables $\{\sigma_i\}$ and $\{\tau_{i-1/2}\}$ on a one-dimensional lattice with periodic boundary conditions, where σ spins occupy lattice sites, and τ spins occupy the links between sites. Then $n_i = \frac{1}{2}(1 + \sigma_i)$ represents the particle occupation of site i, whereas $\tau_{i-1/2} = +1$ or -1 represents the local slope of the surface (denoted / or \, respectively). The dynamics of the interface is that of the single-step model [8], with stochastic corner flips involving exchange of adjacent τ's, thus /\ $\to$ \/ with rate

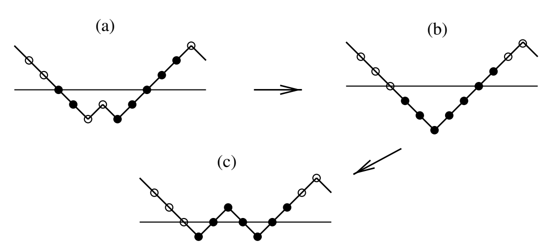

FIG. 1. Depicting clustering of particles (•) in a section of the fluctuating surface. A surface fluctuation (a) → (b) causes the particles to roll into a valley. Particles remain clustered even after a reverse surface fluctuation (b) → (c) occurs.

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p_1, while $\backslash/ \to /\backslash$ with rate q_1. A particle and a hole on adjacent sites $(i, i+1)$ exchange with rates that depend on the intervening local slope $\tau_{i-1/2}$, thus the moves $\bullet\backslash\circ \to \circ\backslash\bullet$ and $\circ/\bullet \to \bullet/\circ$ occur at rate p_2, while the inverse moves occur with rate $q_2 < p_2$. The asymmetry of the rates reflects the fact that it is easier to move downwards along the gravitational field. Note that the dynamics conserves $\sum\sigma$ and $\sum\tau$; we work in the sector where both vanish. In the remainder of this paper, we report results for symmetric surface fluctuations ($p_1 = q_1$), whose behavior at large length and time scale is described by the continuum Edwards-Wilkinson model [9]. Further we consider the strong-field ($q_2 = 0$) limit for the particle system, and set $p_2 = p_1$. We have also investigated the behavior away from these limits, and found that our broad conclusions remain unaffected. The SP model is a limiting case of the Lahiri-Ramaswamy model of sedimenting colloidal crystals [4]; it corresponds to the tilt field evolving autonomously.

In the SP model, particles preferentially occupy the lower portions or large valleys of the fluctuating surface. In order to study the dynamics of the hills and valleys of the surface, we define a height profile $\{h_i\}$ with $h_i = \sum_{1\leq j\leq i}\tau_{j-1/2}$. We then define a coarse-grained depth (CD) model by considering spins $s_i = -\mathrm{sgn}(h_i)$, where s_i is $+1$, -1, or 0 if the surface height h_i at site i is below, above, or at the zero level. A stretch of like s_i's $= +1$ represents a valley with respect to the zero level. The time evolution of the CD model variables $\{s_i\}$ is induced by the underlying dynamics of the bond variables $\{\tau_{i-1/2}\}$. The model is similar to the domain growth model of Kim *et al.* [10].

In our numerical simulations, we studied the evolution of the density in the SP model starting from an initial random placement of particles on the fluctuating surface. In every Monte Carlo step, we performed $2L$ random sequential updates of the site and bond variables. After an initial quick downward slide into local valleys, the density distribution is guided by the evolution of the surface profile. To quantify the tendency towards clustering, we monitored the equal time correlation function $C(r,t) \equiv \langle\sigma_i(t)\sigma_{i+r}(t)\rangle$ (Fig. 2). If z is the dynamical exponent characteristic of the surface fluctuations ($z = 2$ for the symmetric surface model), we expect the scale $\mathcal{L}(t)$ for density fluctuations to be set by the base lengths of typical coarse-grained hills which have overturned in time t, i.e. $\mathcal{L}(t) \sim t^{1/z}$. This is indeed the case in the scaling limit [$r \gg 1$, $t \gg 1$, $r/\mathcal{L}(t)$ fixed] as shown by the collapse to a scaling function $C_s(y = r/\mathcal{L}(t))$ in Fig. 2. We found similar scaling collapses for other models of surface fluctuations with widely different values of z [$z = 3/2$ for Kardar-Parisi-Zhang (KPZ) [11] surfaces, and $z \simeq 4$ for the Das Sarma–Tamborenea model [12]].

The existence of a single growing length scale $\mathcal{L}(t)$ is indicative of coarsening towards a phase-ordered state [6]. In the SP model, coarsening is driven by surface fluctuations, rather than more customary temperature quenches.

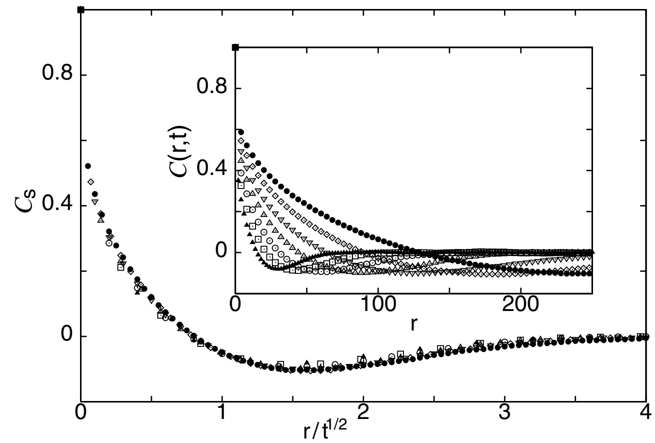

FIG. 2. The data for $C(r,t)$ at different times $t = 400 \times 2^n$ (with $n = 0, \ldots, 6$), shown in the inset, is shown to collapse when scaled by $\mathcal{L}(t) \sim t^{1/2}$.

This causes an interesting feature to appear in the scaling function C_s, namely, a distinctive cusp for small argument: $C_s(y) = C_0 - C_1 y^\alpha$ for $y \ll 1$ (Fig. 2). We find the cusp exponent $\alpha \simeq 0.5$. This cusp implies that the scaled structure factor $S \sim (k\mathcal{L})^{-(1+\alpha)}$ for large $k\mathcal{L}$. This is substantially different from the Porod law behavior $(k\mathcal{L})^{-2}$, characteristic of customary coarsening systems [6]. The cusp is also present in the steady state correlation function, and its origin is discussed below.

In the steady state, we monitored the magnitude of the Fourier components of the density profile

$$Q(k) = \left|\frac{1}{L}\sum_{j=1}^{L} e^{ikj} n_j\right|, \qquad k = \frac{2\pi m}{L}, \tag{1}$$

where $m = 1, \ldots, L-1$. As in [13] we used the lowest nonzero Fourier component $Q^* \equiv Q(\frac{2\pi}{L})$ as a measure of the phase separation in our system with conserved dynamics. The time average $\langle Q^*\rangle \simeq 0$ in a disordered state, and is $\simeq 0.318$ in a fully phase-separated state. For the SP model, Fig. 3 shows numerical results for $\langle Q(k)\rangle$ versus k for various system sizes. While $\langle Q^*\rangle$ approaches a finite limit as $L \to \infty$, the values of Fourier components at fixed k decrease with increasing L. This provides strong evidence for phase separation, corresponding to the occurrence of density inhomogeneities of the order of the system size. However $Q^*(t)$ shows strong fluctuations as a function of time (Fig. 3, inset). With increasing L, the separations between fluctuations increase with L, but the amplitude of fluctuations does not decrease, reminiscent of the behavior of the order parameter in a model of competitive learning [14]. The best way to characterize these strong fluctuations is through the probability distribution $\mathrm{Prob}(Q^*)$. We found numerically that $\mathrm{Prob}(Q^*)$ is non-Gaussian and is characterized by a mean value $\langle Q^*\rangle \simeq 0.18$, and rms fluctuation $(\langle Q^{*2}\rangle - \langle Q^*\rangle^2)^{1/2} \simeq 0.07$. Despite these large fluctuations in Q^*, the configuration of the system does not become randomly disordered even when $Q^*(t)$ is small. Rather, the next few Fourier modes

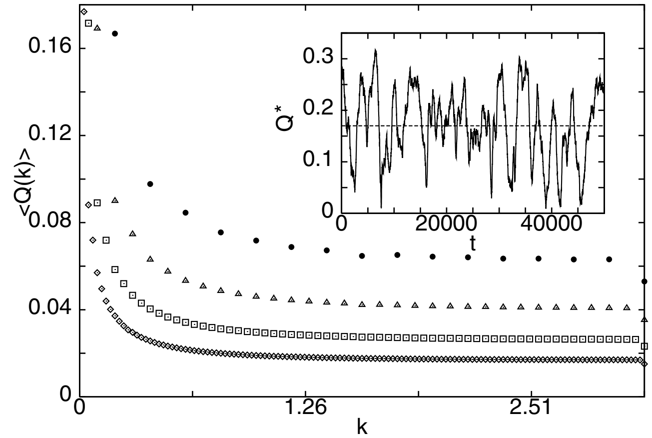

FIG. 3. $\langle Q(k)\rangle$ as a function of k, for different sizes $L = 32$, 64, 128, and 256 (from above to below). Inset: time variation of Q^* in the steady state for $L = 128$.

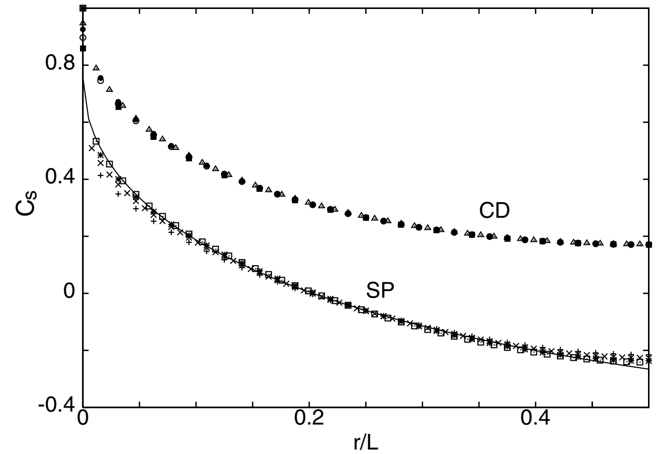

FIG. 4. $C(r)$ in the steady state of the SP and CD models for $L = 64$, 128, 256, and 512. Notice the cusp at small $|r|/L$. The curves are fits to the form $c_0 - c_1 y^{1/2} + c_2 y$.

($k = 4\pi/L, 6\pi/L, \ldots$) are excited at such times, indicating a breakup into a few macroscopic regions of different density.

Not surprisingly, these macroscopic fluctuations leave a strong imprint on spatial correlation functions and cluster distributions. For instance, the two-point correlation function $C(r) \equiv \langle \sigma_j \sigma_{j+r}\rangle$ varies with r on the scale of the system size L, as is evident from Fig. 4 which shows that the data collapse onto a single curve when plotted versus r/L. For comparison, recall that a phase-separated system with sharp interfaces between two macroscopic phases would show a linear decrease $C(r) = M_0^2(1 - 2|r|/L)$ on length scales larger than the correlation length. By contrast, the curve in Fig. 4 is nonlinear in the full range of r/L. In particular, like C_s in the coarsening regime, the scaling function C_s for steady state correlations has a cusp for small argument:

$$C_s\left(\frac{r}{L}\right) = c_0 - c_1\left|\frac{r}{L}\right|^{\alpha}, \qquad \left|\frac{r}{L}\right| \ll 1, \tag{2}$$

with $\alpha \simeq 0.5$. Further, we found that in the steady state the size distribution of clusters (groups of contiguous lattice sites occupied by like σ's) follows a power law $P(l) \sim l^{-\theta}$ with $\theta \simeq 1.8$ (Fig. 5). This observation gives a pointer to the physical origin of the cusp in C_s. Interfacial regions between particle-rich and particle-poor phases contain several particle and hole clusters drawn from $P(l)$; the resulting structure of these regions is responsible for the non-Porod behavior of C_s.

There is a relationship between the cusp exponent α and the cluster power law exponent θ within the independent interval approximation (IIA) [7]. Within this scheme, the joint probability of having n successive intervals is treated as the product of the distribution of single intervals. In our case, the intervals are successive clusters of particles and holes, which occur with probability $P(l)$. Defining the Laplace transform $\tilde{P}(s) = \int_0^\infty dl\, e^{-ls} P(l)$, and $\tilde{C}(s)$ analogously, we have [7]

$$s(1 - s\tilde{C}(s)) = \frac{2}{\langle l\rangle}\frac{1 - \tilde{P}(s)}{1 + \tilde{P}(s)}, \tag{3}$$

where $\langle l\rangle$ is the mean cluster size. In usual applications of the IIA, the interval distribution $P(l)$ has a finite first moment $\langle l\rangle$ independent of L. But that is not the case here, as $P(l)$ decays as a slow power law $P(l) \sim l^{-\theta}$ for $l \gg 1$. Since the largest possible value of l is L, we have $\langle l\rangle \approx aL^{2-\theta}$ for large enough L. Considering s in the range $1/L \ll s \ll 1$, we may expand $\tilde{P}(s) \approx 1 - bs^{\theta-1}$; then to leading order the right hand side of Eq. (3) becomes $bs^{\theta-1}/aL^{2-\theta}$, implying $\tilde{C}(s) \approx 1/s - b/(aL^{2-\theta}s^{3-\theta})$. This leads to

$$C(r) \approx 1 - \frac{b}{a\Gamma(3-\theta)}\left|\frac{r}{L}\right|^{2-\theta}. \tag{4}$$

This has the same scaling form as Eq. (2). Matching the cusp singularity in Eqs. (2) and (4), we get $\theta + \alpha = 2$. By comparison, the numerically determined values for the SP model yield $\theta + \alpha \simeq 2.3$. We conclude that the IIA provides useful insights, even though it is not exact.

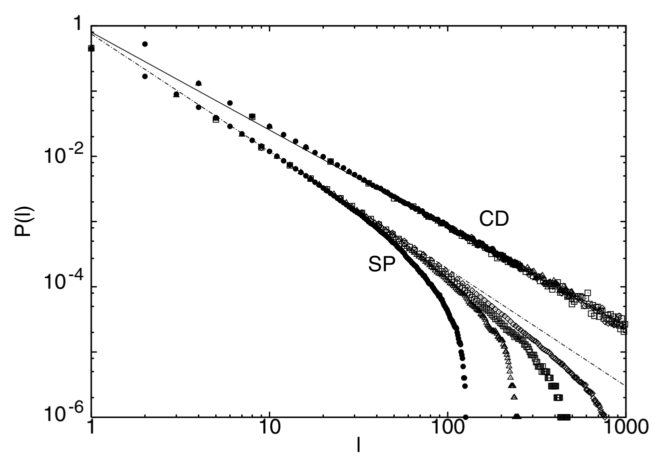

FIG. 5. Probability distribution $P(l)$ of particle cluster lengths for the SP and CD models for $L = 256$, 512, 2024, and 2048. The straight lines have slopes -1.8 and -1.5, respectively.

We can gain considerable understanding into the nature of phase ordering in the SP model by analyzing the closely related CD model of surface fluctuations. The relationship between these models, which is plausible at a qualitative level, can be quantified by checking whether the overlap $O = \langle \sigma_i s_i \rangle$ is nonzero; numerically we find $O \simeq 0.26$. The similarity in the behavior of the two models is brought out in Figs. 4 and 5; $C(r/L)$ has a cusp and $P(l)$ has a power law decay, for both models.

We now show that the CD model can be mapped onto a random walk (RW) problem and that the mapping can be exploited to give exact results for several properties. We make a correspondence between each surface configuration and a RW trajectory, by interpreting $\tau_{i-1/2} = +1$ or -1 as the rightward or leftward RW step at the ith time instant. Then $s_i = 1, -1$, or 0 depending upon whether the walker is to the right, to the left, or at the origin after the ith step. Evidently, the lengths of clusters of $s = 1$ spins (or $s = -1$ spins) represent times between successive returns to the origin. Thus, $P(l)$ for the CD model is just the well-known distribution for RW return times to the origin, which behaves as $l^{-3/2}$ for large l, implying $\theta = 3/2$. Since successive RW returns are independent events, the IIA is exact for the CD model. Hence $\theta + \alpha = 2$, and we conclude that $\alpha = 1/2$.

In systems with strong macroscopic fluctuations such as the SP and CD models, characterization of the steady state requires the full probability distribution of the order parameter. For the CD model, an appropriate (nonconserved) order parameter is $M = \frac{1}{L}\sum s_i$, which for the RW represents the excess time a walker spends on one side of the origin over the other side. In order to respect periodic boundary conditions, we need to restrict the ensemble of RWs to those which return to the origin after L steps. The full probability distribution of M over this ensemble is known from the equidistribution theorem on sojourn times of a RW [15]: $\text{Prob}(M) = 1/2$ for $M \in [-1, 1]$, i.e., every allowed value of M is equally likely. This implies $\langle |M| \rangle = 1/2$ and $(\langle M^2 \rangle - \langle |M| \rangle^2)^{1/2} = 1/\sqrt{12}$. The strong fluctuations in M mirror the large fluctuations of Q^* in the SP model.

To summarize, both the SP and CD models exhibit a phase-ordered steady state with unusual fluctuation characteristics, manifested in several related ways: slow power law decays of the cluster size distribution, a cusp singularity in the scaled two-point correlation function, and a probability distribution for the order parameter which remains broad even in the thermodynamic limit.

We have checked numerically that this type of state survives in the SP model even when we depart from the strong-field limit by allowing q_2 to be nonzero, and when we vary the ratio p_2/p_1 of rates of particle and surface motion from high to low values. Finally, we found that even when we allow different rates $p_1 = 1$ and $q_1 = 0$ for upward and downward corner flips, which makes surface fluctuations KPZ-like, this type of phase separation persists and the scaled correlation function shows a cusp [16]. We found that the state is destroyed, however, if there is an overall tilt of the KPZ surface. The state is also unstable under a change of the dynamical rules which allows the σ variables to influence the evolution of the τ's; then, depending on the coupling, there is either a homogeneous wave carrying state or one which is strongly phase separated [4].

It would be interesting to investigate the long-time dynamics of the large-scale fluctuations in these models, and to characterize the steady state in higher dimensions.

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